Quadratic Convex Reformulation for discrete quadratic optimization

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April 2016

Joint work with A. Billionnet, A. Lambert, and M-C. Plateau

Outline

I- Classical approaches for binary quadratic programs

II- Quadratic Convex Reformulation for binary quadratic programs

III- Quadratic Convex Reformulation for quadratically constrained mixed integer quadratic programs

I- Binary quadratic programs -

Classical approaches

Binary quadratic programs

QP01

$$\min \quad f(x) = \sum_i c_i x_i + \sum_i \sum_{j \neq i} q_{ij} x_i x_j \quad = c^T x + x^T Q x$$

$$s.t.:$$

$$Ax = b$$

$$x \in \{0,1\}^n$$

Two main approaches

1- Linearization (or linear reformulation)

2- SemiDefinite Programming relaxation

1- Linearization (or linear reformulation)

Fortet, 1958

- replace $x_i x_j$ by y_{ij} and

$$\begin{cases} y_{ij} \le x_i \\ y_{ij} \le x_j \\ y_{ij} \ge x_i + x_j - 1 \\ y_{ij} \ge 0 \end{cases}$$

1- Linearization (or linear reformulation)

MILP01

$$\min \quad f(x) = \sum_i c_i x_i + \sum_i \sum_{j \neq i} q_{ij} y_{ij}$$

$s.t.:$

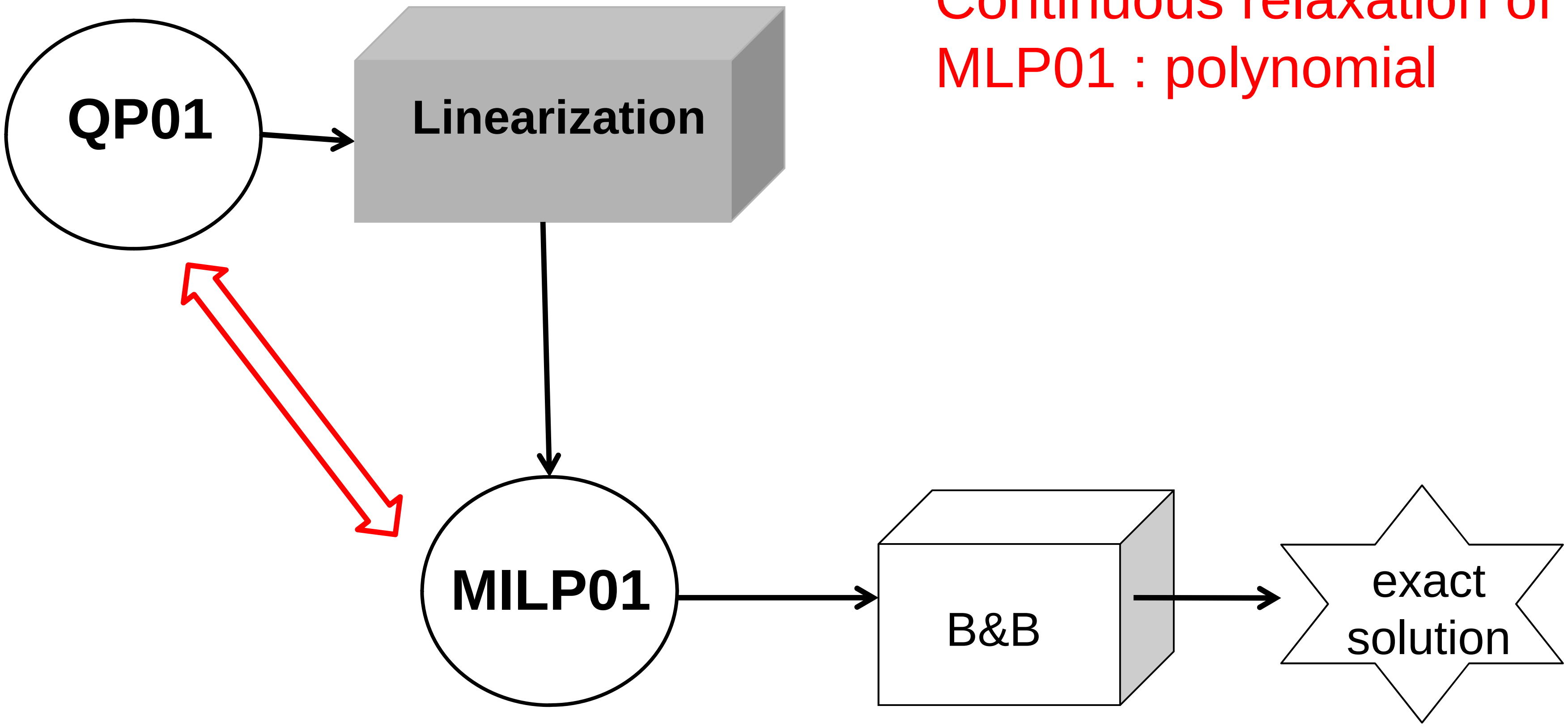
$$Ax = b$$

$$y_{ij} \leq x_i ; y_{ij} \leq x_j ; y_{ij} \geq x_i + x_j - 1; \; y_{ij} \geq 0$$

$$x \in \{0,1\}^n$$

1- Linearization (or linear reformulation)

Continuous relaxation of MLP01 : polynomial



at each node: LP

1- Linearization (or linear reformulation)

Drawback : weak LP bound

(unless very few quadratic terms)

2- SemiDefinite Programming relaxation

-replace xx^T ($x_i x_j$) by matrix X (X_{ij})

relax $X=xx^T$ and $x \in \{0,1\}^n$ by :

$$X_{ii} = x_i \quad \forall i$$

$$\begin{pmatrix} 1 & x^T \\ x & X \end{pmatrix} \succ 0$$

(SDP1):

$$\begin{aligned} &\min \quad c^T x + \langle Q, X \rangle \\ &s.t.: \\ &\qquad Ax = b \\ &\qquad X_{ii} = x_i \\ &\qquad \begin{pmatrix} 1 & x^T \\ x & X \end{pmatrix} \succ 0 \end{aligned}$$

2- SemiDefinite Programming relaxation

-The SDP bound can be used in a B&B algorithm

-sharp bounds

-drawback : time (at each node)

2- SemiDefinite Programming relaxation

- 1st improvment

(SDP2):

$$\min \quad \sum_{i=1}^{n} c_i x_i + \sum_{i,j} q_{ij} X_{ij}$$

$$s.t.:$$

$$Ax = b$$

$$X_{ii} = x_i$$

$$\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ki} a_{kj} X_{ij} = b_k^2 \quad \forall k$$

$$\begin{pmatrix} 1 & x^T \\ x & X \end{pmatrix} \succ 0$$

2- SemiDefinite Programming relaxation

- 2nd improvment « **SDP+RLT** » relaxation

(SDP3):

$$\min \quad \sum_{i=1}^{n} c_i x_i + \sum_{i,j} q_{ij} X_{ij}$$

$$s.t.:$$

$$Ax = b$$

$$X_{ii} = x_i$$

$$\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ki} a_{kj} X_{ij} = b_k^2 \quad \forall k$$

$$X_{ij} \le x_i \,, X_{ij} \le x_j \,, X_{ij} \ge x_i + x_j - 1 \,, X_{ij} \ge 0$$

$$\begin{pmatrix} 1 & x^T \\ x & X \end{pmatrix} \succ 0$$

Example

$$
\begin{aligned}
\min f(x) = &-9x_1 - 7x_2 + 2x_3 - 80x_4 + 12x_5 - 48x_1x_2 + 4x_1x_3 + \\
&36x_1x_4 - 24x_1x_5 - 7x_2x_3 + 36x_2x_4 - 84x_2x_5 + 40x_3x_4 + 4x_3x_5 - 88x_4x_5 \\
s.t.& \\
&x_1 + x_2 + 2x_4 + x_5 = 2 \quad x \text{ binary}
\end{aligned}
$$

$$
c = \begin{pmatrix} -9 \\ -7 \\ 2 \\ 23 \\ 12 \end{pmatrix} \qquad Q = \begin{pmatrix} 0 & -24 & 2 & 18 & -12 \\ -24 & 0 & -3.5 & 18 & -42 \\ 2 & -3.5 & 0 & 20 & 2 \\ 18 & 18 & 20 & 0 & -44 \\ -12 & -42 & 2 & -44 & 0 \end{pmatrix}
$$

Example : Linearization

$$\min f(x) = -9x_1 - 7x_2 + 2x_3 - 80x_4 + 12x_5 - 48y_{12} + 4y_{13} +$$
$$36y_{14} - 24y_{15} - 7y_{23} + 36y_{24} - 84y_{25} + 40y_{34} + 4y_{35} - 88y_{45}$$

$s.t.$

$$x_1 + x_2 + 2x_4 + x_5 = 2$$

$$y_{ij} \geq 0 \quad y_{ij} \leq x_i \quad y_{ij} \leq x_j \quad y_{ij} \geq x_i + x_j - 1$$

$$x_{ij} \in \{0,1\}$$

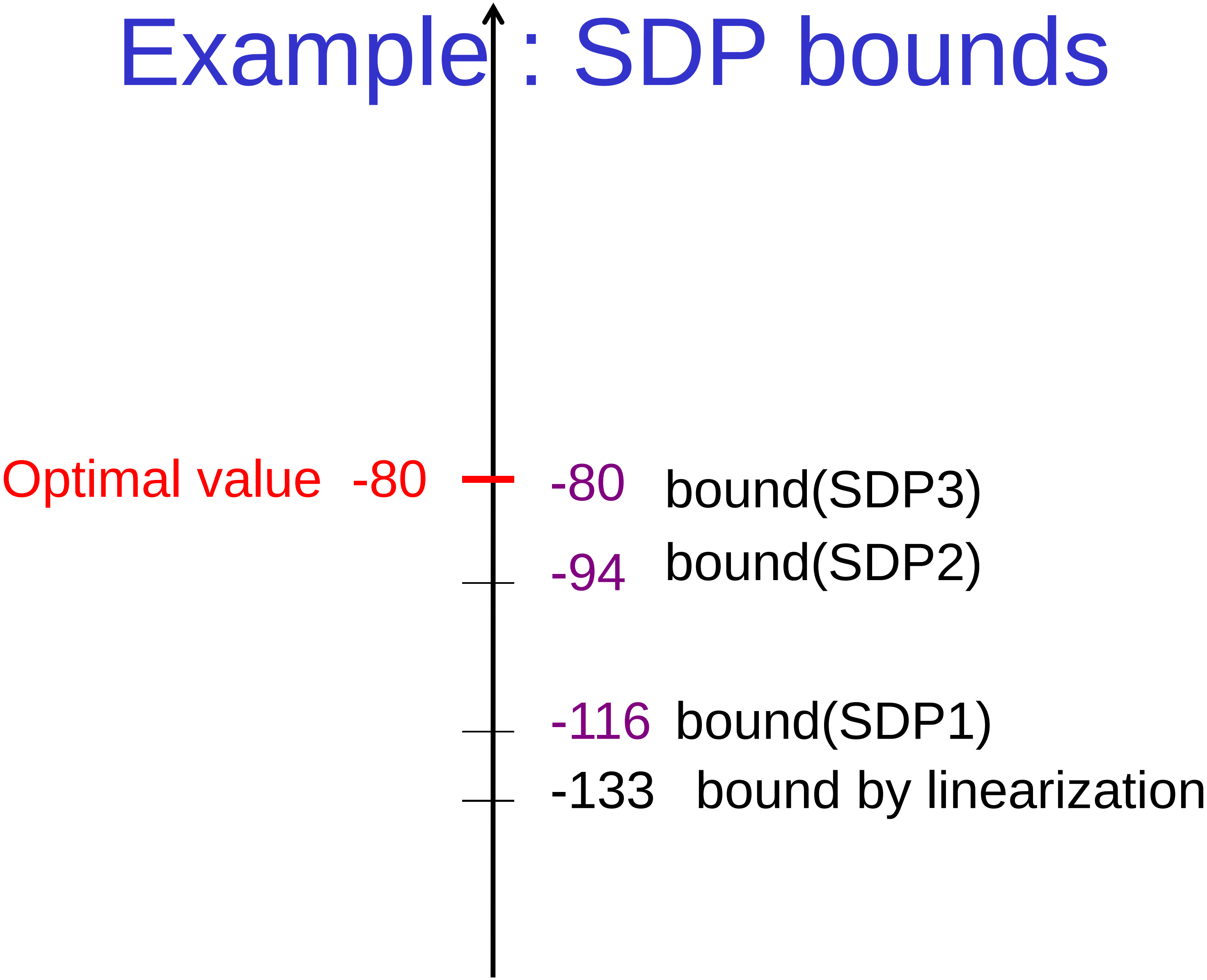
Example : Linearization

```
            Nodes                                            Cuts/
   Node   Left      Objective  IInf  Best Integer      Best Node    ItCnt     Gap

*     0+     0                         -65.0000                         0     ---
      0      0    -133.2000     5      -65.0000     -133.2000        13  104.92%
*     0+     0                         -80.0000     -133.2000        13   66.50%
      0      0       cutoff            -80.0000     -133.2000        15   66.50%

Times (seconds):
Input =  0.005
Solve =  0.004
Output = 0.001
CPLEX 11.2.0: optimal integer solution; objective -80
15 MIP simplex iterations
0 branch-and-bound nodes
```

Example : SDP bounds



Outline

I- Classical approaches for binary quadratic programs

II- Quadratic Convex Reformulation for binary quadratic programs

III- Quadratic Convex Reformulation for quadratically constrained mixed integer quadratic programs

Difference / classical approaches

based on : **quadratic convex optimization**

if the objective is convex: MI**L**P
« generalises » to MI**Q**P

Difference / classical approaches

based on : **quadratic convex optimization**

QP01

$$\min \quad f(x) = \sum_i c_i x_i + \sum_i \sum_{j \neq i} q_{ij} x_i x_j \qquad = c^T x + x^T Q x$$

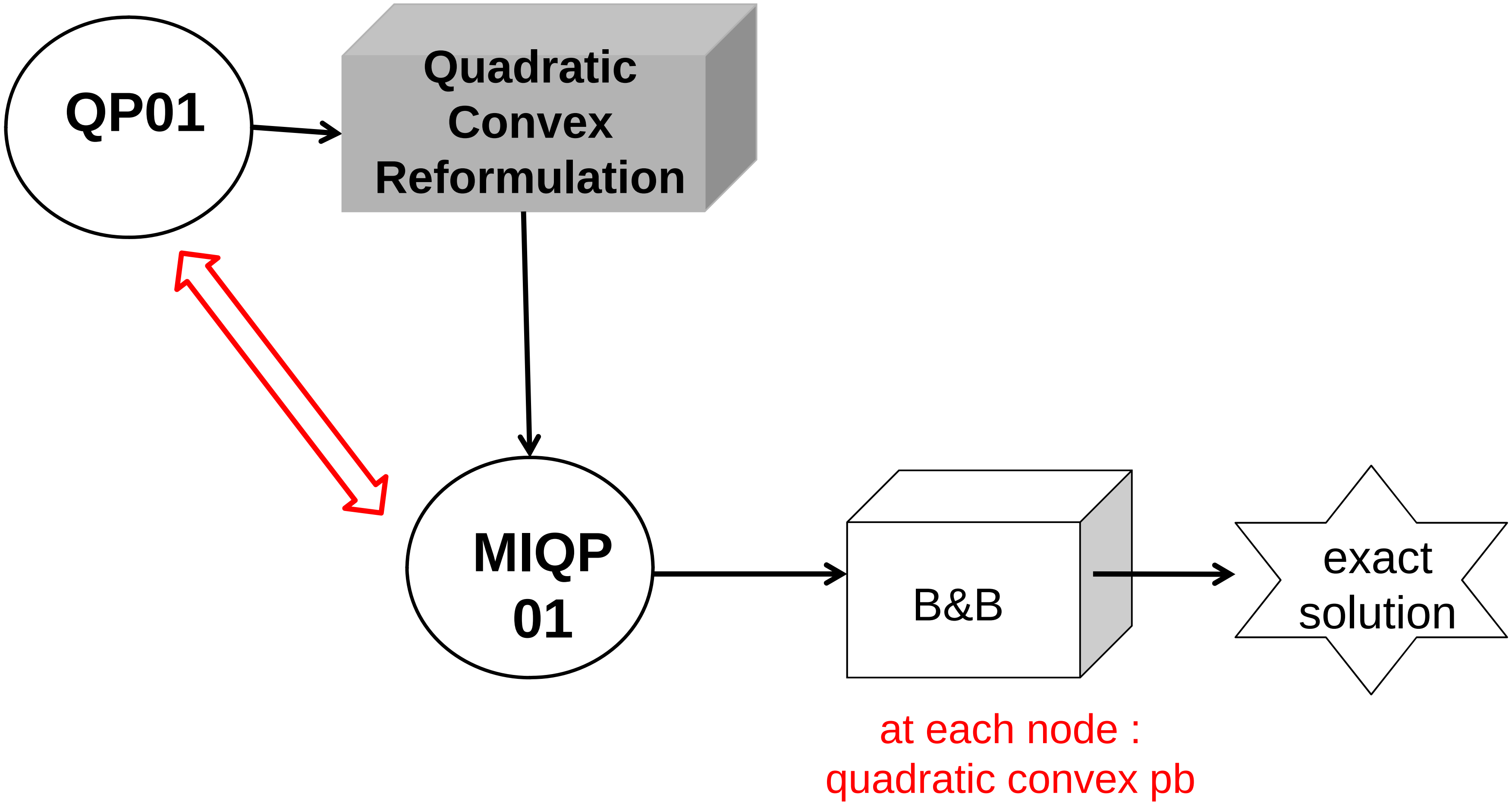
$s.t.:$

$$Ax = b$$

$$x \in \{0,1\}^n$$

if f convex, (Q is psd) then B&B (continuous relaxation)

Quadratic Convex Reformulation



Quadratic Convex Reformulation

Good news : this is possible for any binary quadratic program

« Direct » Quadratic Convex Reformulation

$$q_{ij}x_ix_j = \frac{1}{2}q_{ij}\left[(x_i + x_j)^2 - x_i^2 - x_j^2\right]$$ if $q_{ij} > 0$

$$= \frac{1}{2}q_{ij}\left[(x_i + x_j)^2 - x_i - x_j\right]$$ since $x_i^2 = x_i$

$$q_{ij}x_ix_j = \frac{1}{2}(-q_{ij})\left[(x_i - x_j)^2 - x_i - x_j\right]$$ if $q_{ij} < 0$

« Direct » Quadratic Convex Reformulation

$$\min f(x) = -9x_1 - 7x_2 + 2x_3 - 80x_4 + 12x_5 - 48x_1x_2 + 4x_1x_3 + \\ 36x_1x_4 - 24x_1x_5 - 7x_2x_3 + 36x_2x_4 - 84x_2x_5 + 40x_3x_4 + 4x_3x_5 - 88x_4x_5$$

$s.t.$

$$x_1 + x_2 + 2x_4 + x_5 = 2 \qquad x \text{ binary}$$

Reformulated problem:

$$\min f(x) = -9x_1 - 7x_2 + 2x_3 - 80x_4 + 12x_5 + (24(x_1 - x_2)^2 - x_1 - x_2) + 2...$$

$s.t.$

$$x_1 + x_2 + 2x_4 + x_5 = 2 \qquad x \text{ binary}$$

Amounts to:

$$c = \begin{pmatrix} -9 \\ -7 \\ 2 \\ -80 \\ 12 \end{pmatrix} \qquad Q = \begin{pmatrix} 0 & -24 & 2 & 18 & -12 \\ -24 & 0 & -3.5 & 18 & -42 \\ 2 & -3.5 & 0 & 20 & 2 \\ 18 & 18 & 20 & 0 & -44 \\ -12 & -42 & 2 & -44 & 0 \end{pmatrix}$$

$$c = \begin{pmatrix} -9-24 \\ -7-24 \\ 2 \\ -80 \\ 12 \end{pmatrix} \qquad Q = \begin{pmatrix} +24 & -24 & 2 & 18 & -12 \\ -24 & +24 & -3.5 & 18 & -42 \\ 2 & -3.5 & 0 & 20 & 2 \\ 18 & 18 & 20 & 0 & -44 \\ -12 & -42 & 2 & -44 & 0 \end{pmatrix}$$

« Direct » Quadratic Convex Reformulation

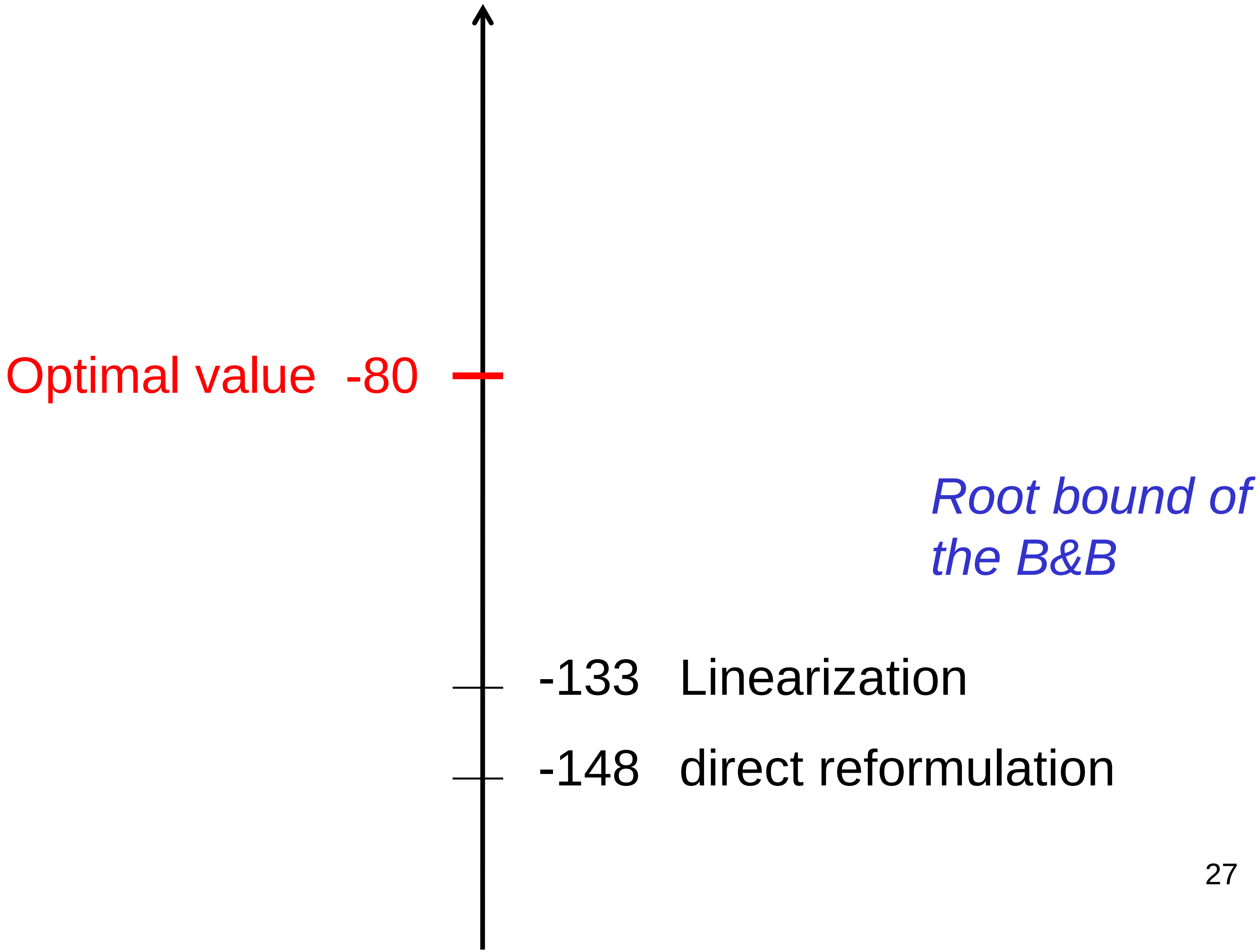
```
 Nodes                                              Cuts/
   Node  Left     Objective  IInf  Best Integer     Best Node    ItCnt     Gap

*     0+    0                          -79.0000                     0     ---
      0     0     -148.4014     5      -79.0000     -148.4014       7   87.85%
*     0+    0                          -80.0000     -148.4014       7   85.50%
      0     1     -133.7123     4      -80.0000     -107.8295       8   34.79%
      1     1     -107.8295     3      -80.0000      -86.3841       9    7.98%
      2     0      -86.3841     1      -80.0000      -80.0000      10    0.00%

Times (seconds):
Input =  0.006
Solve =  0.006
Output = 0
CPLEX 11.2.0: optimal integer solution; objective -80
10 MIP simplex iterations
3 branch-and-bound nodes
```

At each node: a quadratic convex problem is solved

« Direct » Quadratic Convex Reformulation



Quadratic Convex Reformulation with the smallest eigenvalue

$$\sum_i \sum_j q_{ij} x_i x_j = \sum_i \sum_j q_{ij} x_i x_j - \lambda_{\min} \sum_i \left(x_i^2 - x_i \right)$$

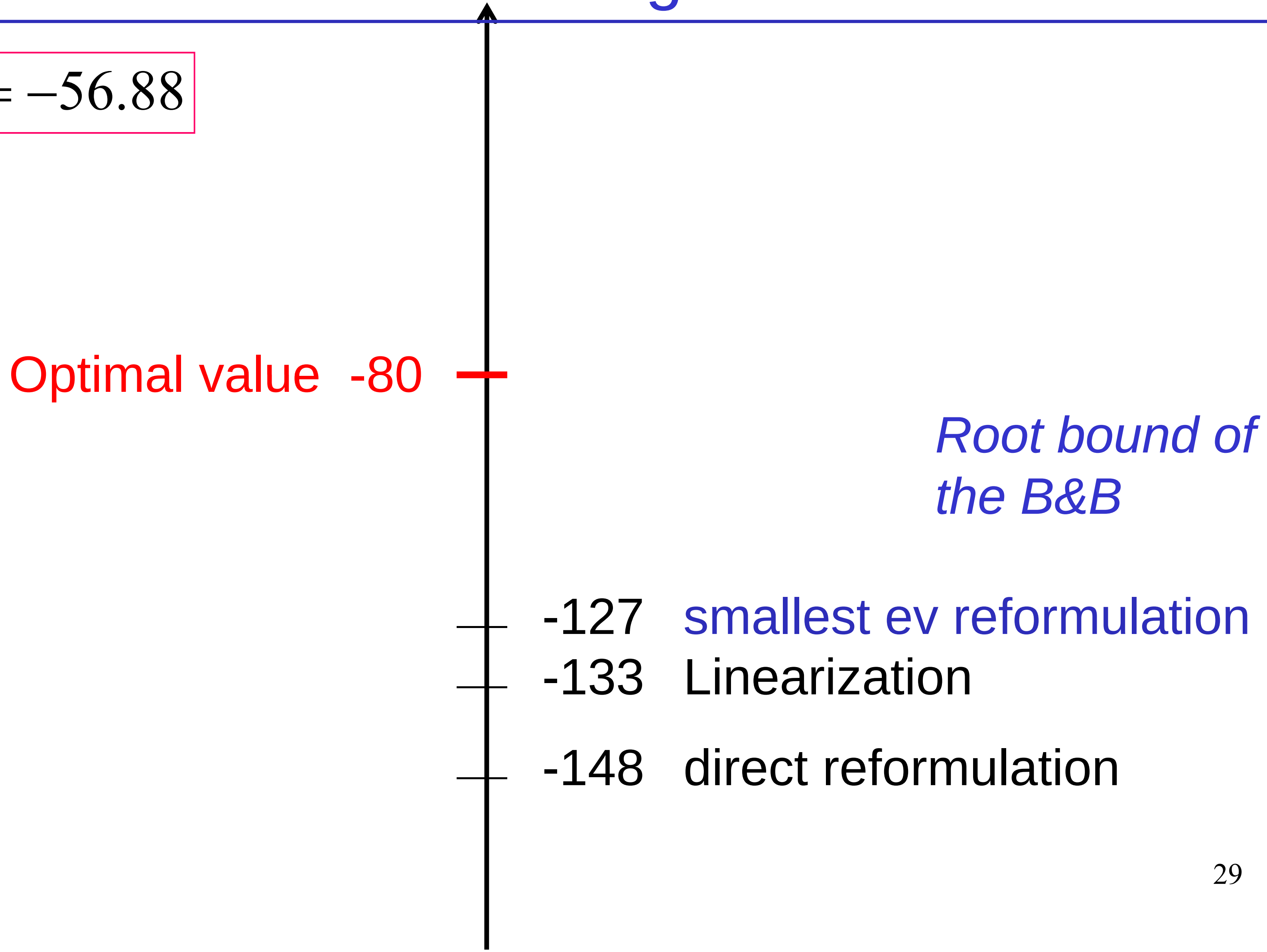
Hammer, Rubin

$\lambda_{\min}$: smallest eigenvalue of Q (<0)

$$c_\lambda = \begin{pmatrix} -9 + \lambda_{\min} \\ -7 + \lambda_{\min} \\ 2 + \lambda_{\min} \\ -80 + \lambda_{\min} \\ 12 + \lambda_{\min} \end{pmatrix} \quad Q_\lambda = \begin{pmatrix} -\lambda_{\min} & -24 & 2 & 18 & -12 \\ -24 & -\lambda_{\min} & -3.5 & 18 & -42 \\ 2 & -3.5 & -\lambda_{\min} & 20 & 2 \\ 18 & 18 & 20 & -\lambda_{\min} & -44 \\ -12 & -42 & 2 & -44 & -\lambda_{\min} \end{pmatrix}$$

Quadratic Convex Reformulation with the smallest eigenvalue

$$\lambda_{\min} = -56.88$$



- how to improve?

$\Leftrightarrow$ find a quadratic convex reformulation with **sharper root bounds**

Improvement 1

Billionnet, E. 2007

- Generalize the diagonal perturbation

$$c_\lambda = \begin{pmatrix} -9-\lambda_1 \\ -7-\lambda_2 \\ 2-\lambda_3 \\ -80-\lambda_4 \\ 12-\lambda_5 \end{pmatrix} \qquad Q_\lambda = \begin{pmatrix} \lambda_1 & -24 & 2 & 18 & -12 \\ -24 & \lambda_2 & -3.5 & 18 & -42 \\ 2 & -3.5 & \lambda_3 & 20 & 2 \\ 18 & 18 & 20 & \lambda_4 & -44 \\ -12 & -42 & 2 & -44 & \lambda_5 \end{pmatrix}$$

$$f_\lambda(x) = f(x) + \sum_{i=1}^{n} \lambda_i \left(x_i^2 - x_i\right)$$

$$= f(x) \quad \forall x \text{ binary}$$

Improvement 1

- any vector λ s.th. Q+diag(λ) psd is candidate
- find vector λ that maximizes the root bound

MIQP01$_\lambda$

$$\begin{aligned} \min \quad & f_\lambda(x) \\ s.t.: \quad & \\ & Ax = b \\ & x \in \{0,1\}^n \end{aligned}$$

P

$$\max_{\lambda : f_\lambda \text{ convex}} \quad \min \quad f_\lambda(x)$$

$$Ax = b$$

$$x \in [0,1]^n$$

Improvement 1

« Sharpest-bound » Theorem : opt(**P**)=opt(SDP1)

(SDP1):

$$
\begin{aligned}
&\min \quad c^T x + \langle Q, X \rangle \\
&s.t.: \\
&\qquad Ax = b \\
&\qquad X_{ii} = x_i \\
&\qquad \begin{pmatrix} 1 & x^T \\ x & X \end{pmatrix} \succ 0
\end{aligned}
$$

Improvement 1

« Sharpest-bound » Theorem : opt(**P**)=opt(SDP1)

(SDP1):

$$\min \quad c^T x + \langle Q, X \rangle$$

$$s.t.:$$

$$Ax = b$$

$$X_{ii} = x_i \qquad \text{dual variables } \lambda_i$$

$$\begin{pmatrix} 1 & x^T \\ x & X \end{pmatrix} \succ 0$$

$$Q + \lambda I \succ 0$$

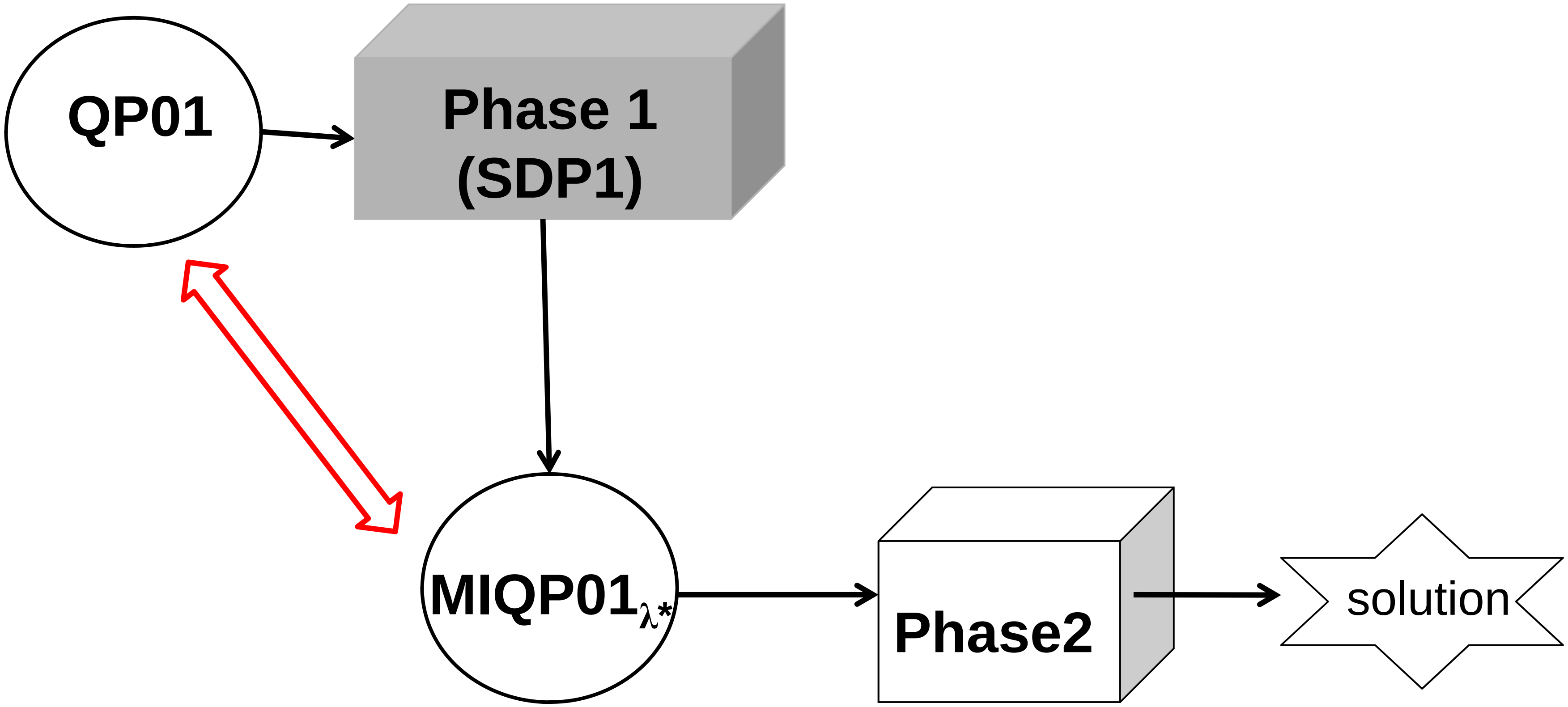
Improvement 1

SDP-based Quadratic Convex Reformulation:

Phase 1 : Solve (SDP1) → vector λ^*

Phase 2 : Solve **MIQP01**$_{\lambda^*}$ by B&B

Root bound: opt(SDP1)


QP01
Phase 1
(SDP1)
MIQP01λ*
Phase2
solution

First results

Unconstrained binary quadratic problem:

$$\min_{x \in \{0,1\}^n} \quad f(x) = \sum_i c_i x_i + \sum_i \sum_{j \neq i} q_{ij} x_i x_j$$

First results

Unconstrained binary quadratic problem:

100 variables, random coefficients in [-100, 100]

Linearization root gap 266% unsolved (1h)

Smallest EV root gap : 15 % unsolved (1h)

QCR root gap: 7%   6 min in average (incl. SDP)

Improvement 2 (QCR)

Billionnet, E., Plateau, 2009

- Use the equality constraints

$$f_{\lambda,\alpha}(x) = f(x) + \sum_{i=1}^{n} \lambda_i \left(x_i^2 - x_i\right) + \sum_k \alpha_k \left(\sum_{j=1}^{n} a_{kj} x_j - b_k \right)^2$$

$$= \alpha b^T b + c_{\lambda,\alpha}^T x + x^T Q_{\lambda,\alpha} x$$

$$f_{\lambda,\alpha}(x) = f(x) \quad \forall x \text{ feasible}$$

Sharpest bound: Similar theorem with:

(SDP2):

$$
\begin{aligned}
\min \quad & \sum_{i=1}^{n} c_i x_i + \sum_{i,j} q_{ij} X_{ij} \\
s.t.: \quad & \\
& Ax = b \\
& X_{ii} = x_i \\
& \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ki} a_{kj} X_{ij} = b_k^2 \quad \forall k \\
& \begin{pmatrix} 1 & x^T \\ x & X \end{pmatrix} \succ 0
\end{aligned}
$$

(λ_i)

(α_k)

Some numerical results

k-cluster or densest k-sub-graph

$$\min \quad f(x) = \sum_i \sum_{j \neq i} w_{ij} x_i (1 - x_j)$$

$$s.t.:$$

$$\sum x_i = k$$

$$x \in \{0,1\}^n$$

Some numerical results

k-cluster or densest k-sub-graph

45 instances with 80 vertices

Smallest EV root gap: 83% unsolved (1h)

QCR root gap: 3.5% solved 40 sec. in average (0.2s for the SDP)

Improvement 3 (MIQCR)

Billionnet, E., Lambert, 2012

- Any change of all entries of Q

$$f_{\lambda,\alpha,\Phi}(x,y) = f(x) + \sum_{i=1}^{n} \lambda_i \left(x_i^2 - x_i\right) + \sum_k \alpha_k \left(\sum_{j=1}^{n} a_{kj} x_j - b_k\right)^2 + \sum_{i,j} \Phi_{ij} \left(x_i x_j - y_{ij}\right)$$

$$= \alpha b^T b + c_{\lambda,\alpha,\Phi}^T x + x^T Q_{\lambda,\alpha,\Phi} x - \Phi y$$

$$\boxed{f_{\lambda,\alpha,\Phi}(x,y) = f(x) \quad \forall x \text{ feasible and } y_{ij} = x_i x_j}$$

- We want: $Q_{\lambda,\alpha,\Phi}$ psd

QP01
(Reform.)

$$\begin{cases} \min \quad f_{\lambda,\alpha,\Phi}(x,y) \\ \text{s.t.}: \quad Ax = b \\ y_{ij} \leq x_i \\ y_{ij} \leq x_j \\ y_{ij} \geq x_i + x_j - 1 \\ y_{ij} \geq 0 \\ x_i \in \{0,1\} \end{cases}$$

$\Rightarrow$ $y_{ij} = x_i x_j$

for $\Phi_{ij} \neq 0$

- maximise the continuous relaxation bound:

(P) :

$$\max_{\lambda,\alpha,\Phi : Q_{\lambda,\alpha,\Phi} \succ 0} \begin{cases} \min & f_{\lambda,\alpha,\Phi}(x,y) \\ \text{s.t.:} & Ax = b \\ & y_{ij} \le x_i \\ & y_{ij} \le x_j \\ & y_{ij} \ge x_i + x_j - 1 \\ & y_{ij} \ge 0 \\ & 0 \le x_i \le 1 \end{cases}$$

Sharpest bound: similar theorem with

(SDP3):

$$\min \quad \sum_{i=1}^{n} c_i x_i + \sum_{i,j} q_{ij} X_{ij}$$

dual var

$s.t.:$

$$X_{ii} = x_i \qquad \forall i \qquad (\lambda_i)$$

$$Ax = b$$

$$\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ki} a_{kj} X_{ij} = b_k^2 \qquad \forall k \qquad (\alpha_k) \quad \text{(QCR)}$$

$$X_{ij} \le x_i \,, X_{ij} \le x_j \,, X_{ij} \ge x_i + x_j - 1 \,, X_{ij} \ge 0 \qquad (\Phi_{ij}) \quad \text{(MIQCR)}$$

$$\begin{pmatrix} 1 & x^T \\ x & X \end{pmatrix} \succ 0$$

Numerical results

k-cluster or densest k-sub-graph

45 instances with 80 vertices

QCR root gap: 3.5% solved in 40 sec. in average (0.2s for SDP)
600 000 nodes

MIQCR root gap: 1% 630 sec. (450 for SDP- 180 for B&B)
6 000 nodes

Numerical results

k-cluster or densest k-sub-graph

Drawback of MIQCR : time for solving (SDP3)

Conic Bundle method applied to SDP : Conic Bundle Library (Helmberg) and CSDP

-> MIQCR-CB (next talk by Amélie Lambert)

Numerical results

k-cluster or densest k-sub-graph

QCR root gap: 3.5% solved in 40 sec. in average (0.2s for SDP)
600 000 nodes

MIQCR root gap: 1% 630 sec. (450 for SDP- 180 for B&B)
6 000 nodes

MIQCR-CB root gap: 1% 10 sec. (6 for SDP- 4 for B&B)

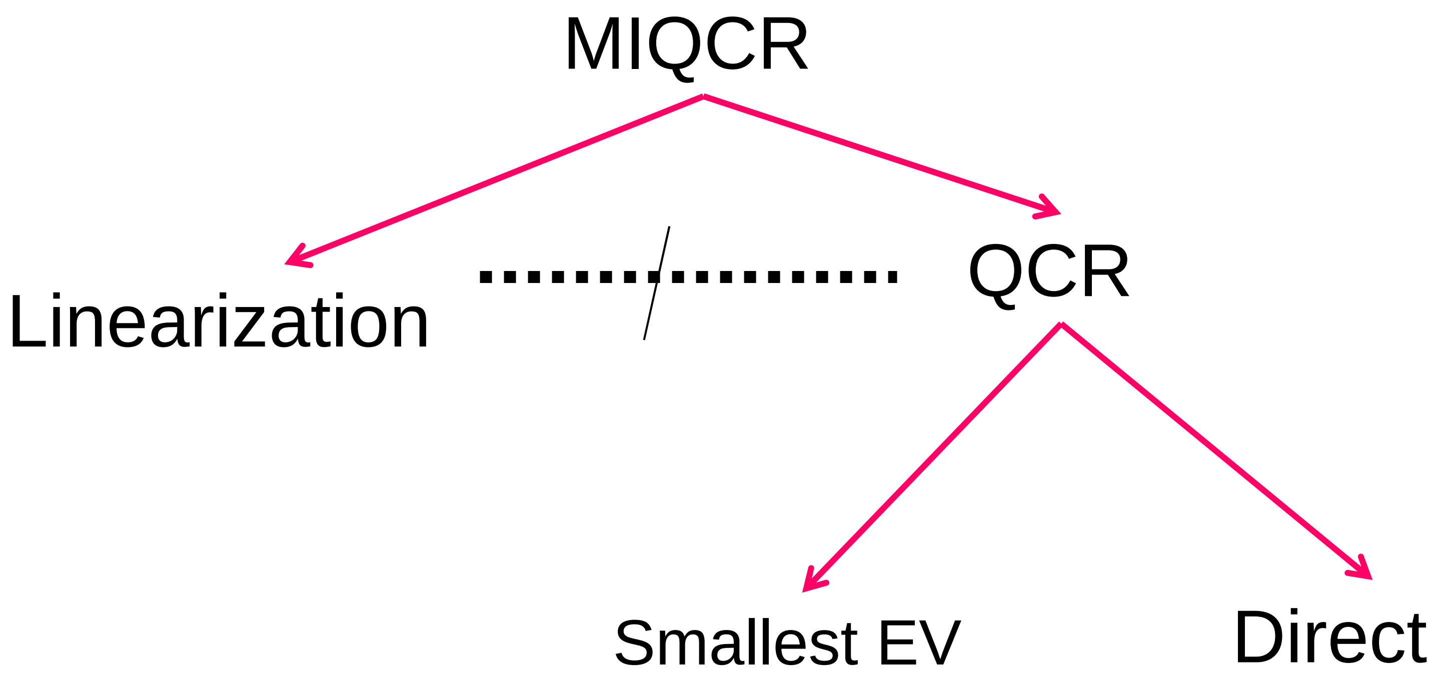
MIQCR-CB instances with up to 160 vertices

To conclude this part:

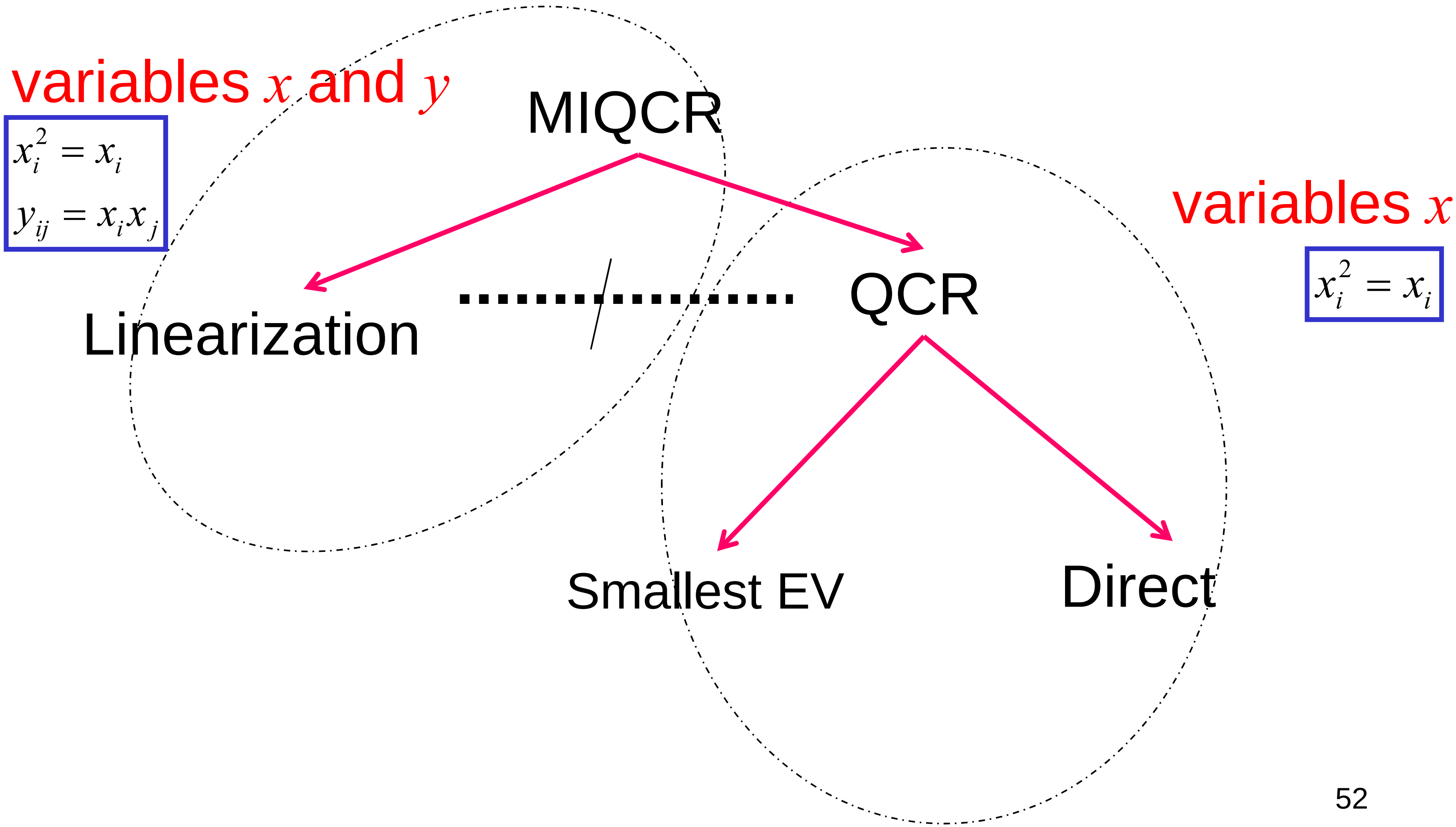
5 **general** methods for exact solution of binary quadratic programs :

Linearization, Direct, Smallest EV, QCR and MIQCR

To conclude this part: hierarchy of root bounds



To conclude this part: hierarchy of root bounds



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Extension to the cases of general integers and quadratic constraints

General bounded integers

$$0 \leq x_i \leq u_i$$

Here, $x_i^2 \neq x_i$

No expression of $y_{ij} = x_i x_j$ by linear constraints

General bounded integers: how to « linearize » $y_{ij} = x_i x_j$?

$$x_i = \sum_{k=0}^{\lfloor \log(u_i) \rfloor} 2^k t_{ik}$$

binary variables

$$y_{ij} = x_i x_j = \sum_{k=0}^{\lfloor \log(u_i) \rfloor} 2^k t_{ik} x_j$$

Linearize with new variables and constraints

$$z_{ijk} = t_{ik} x_j$$

$$z_{ijk} \le x_j \, , z_{ijk} \le u_j t_{ik} \, , z_{ijk} \ge x_j + u_j t_{ik} - u_j \, , z_{ijk} \ge 0$$

General bounded integers: how to linearize $y_{ij} = x_i x_j$?

We get:

$$x_i = \sum_{k=0}^{\lfloor \log(u_i) \rfloor} 2^k t_{ik}$$

$$y_{ij} = \sum_{k=0}^{\lfloor \log(u_i) \rfloor} 2^k z_{ijk}$$

$$z_{ijk} \le x_j, z_{ijk} \le u_j t_{ik}, z_{ijk} \ge x_j + u_j t_{ik} - u_j, z_{ijk} \ge 0$$

$$t_{ik} \in \{0,1\}$$

General bounded integers: how to linearize $y_{ij} = x_i x_j$?

We add strong valid inequalities:

$$y_{ij} = y_{ji}$$

$$y_{ij} \geq u_j x_i + u_i x_j - u_i u_j$$

$$y_{ij} \leq u_j x_i$$

$$y_{ii} \geq x_i$$

$$y_{ij} \geq 0$$

It is a linerization

Finally:

set P_{xyzt}

Linear constraints and binary variables t

General bounded integers: how to linearize $y_{ij} = x_i x_j$?

Lemma

Projection of $\overline{P_{xyzt}}$ on (x,y) is

$$\overline{P_{xy}} \quad \begin{array}{|l|} y_{ij} = y_{ji} \\ y_{ij} \geq u_j x_i + u_i x_j - u_i u_j \\ y_{ij} \leq u_j x_i \\ y_{ii} \geq x_i \\ y_{ij} \geq 0 \end{array}$$

General bounded integers: how to linearize $y_{ij} = x_i x_j$?

Just remember:

- This is possible
- Easy projection on (x,y)

Quadratic Convex Reformulation with general integers and quadratic constraints

QCQP

$$
\begin{aligned}
&\min \quad f(x) = x^T Q_0 x + c_0^T x \\
&s.t.: \\
&\qquad c_k^T x + x^T Q_k x \le b_k \qquad k = 1,\dots,K \\
&\qquad 0 \le x \le u \\
&\qquad x \ \text{integer}
\end{aligned}
$$

Quadratic Convex Reformulation with general integers and quadratic constraints

Let S be a **psd** matrix, write:

$$q_{ij}x_i x_j = s_{ij}x_i x_j + \left(q_{ij} - s_{ij}\right)y_{ij}$$

$$x^T Q x \quad = x^T S x + \langle Q - S, y \rangle$$

Same idea but a more general formalism than in the 0-1 variables case

Quadratic Convex Reformulation with general integers and quadratic constraints

-add variables y: $y_{ij}=x_i x_j$ (or $y=xx^T$)

-Let $S_0, S_1, \ldots, S_K$ *be* ***any*** psd matrices

$$
\begin{array}{ll}
\min & f(x,y) = x^T S_0 x + c_0^T x + < Q_0 - S_0, y > \\
s.t.: & \\
 & x^T S_k x + c_k^T x + < Q_k - S_k, y > \leq b_k \qquad k = 1,\ldots,K \\
 & 0 \leq x \leq u \\
 & x \ \text{integer} \\
 & y = xx^T
\end{array}
$$

« to be linearized »

Quadratic Convex Reformulation with general integers and quadratic constraints

$$
\begin{aligned}
\min \quad & f(x,y) = x^T S_0 x + c_0^T x + < Q_0 - S_0, y > \\
s.t.: & \\
& x^T S_k x + c_k^T x + < Q_k - S_k, y > \leq b_k \qquad k = 1,\dots,K \\
& 0 \leq x \leq u \\
& x \ \text{integer} \\
& x, y, z, t \in P_{xyzt}
\end{aligned}
$$

Equivalent problem
Solved by B&B

Quadratic Convex Reformulation with general integers and quadratic constraints

Particular psd matrices :

- Complete linearization: all S_k=0

-Smallest EV : $S_k = Q_k - diag(\lambda_{\min}(Q_k))$

- Already convex: S_k=Q_k

Quadratic Convex Reformulation with general integers and quadratic constraints

Problem

Find the set of psd matrices

That would lead to the sharpest bound by continuous relaxation

Quadratic Convex Reformulation with general integers and quadratic constraints

Problem

max
$S_0, S_1, \ldots, S_K$
SDP

$$
\begin{aligned}
&\min \quad f(x,y) = x^T S_0 x + c_0^T x + < Q_0 - S_0, y > \\
&s.t.: \\
&\qquad x^T S_k x + c_k^T x + < Q_k - S_k, y > \leq b_k \qquad k = 1,\ldots,K \\
&\qquad 0 \leq x \leq u \\
&\qquad x \ \text{ entier} \\
&\qquad x, y, z, t \in \overline{P_{xyzt}}
\end{aligned}
$$

can be replaced by

$$x, y \in \overline{P_{xy}}$$

Theorem

Same optimal value as:

(SDP)

$$\min \quad c_0^T x + Q_0 X$$

$$s.t.:$$

$$c_k^T x + Q_k X \le b_k \qquad (\alpha_k)$$

$$X_{ij} \le u_j x_i \ , X_{ij} \le u_i x_j \ ,$$

$$X_{ij} \ge u_j x_i + u_i x_j - u_i u_j \ , X_{ii} \ge x_i, \qquad (\Phi_{ij})$$

$$X_{ij} \ge 0$$

$$\begin{pmatrix} 1 & x^T \\ x & X \end{pmatrix} \succ 0$$

dual var

SDP relaxation of (QCQP)

Theorem-

deduce:

$$S_0^* = Q_0 + \sum_k \alpha_k Q_k + \Phi$$

$$S_k^* = 0 \qquad \text{for } k \geq 1$$

as « optimal » matrices

Quadratic Convex Reformulation with general integers and quadratic constraints

Hence, in the optimal reformulation :

Objective function: quadratic

Constraints: linear

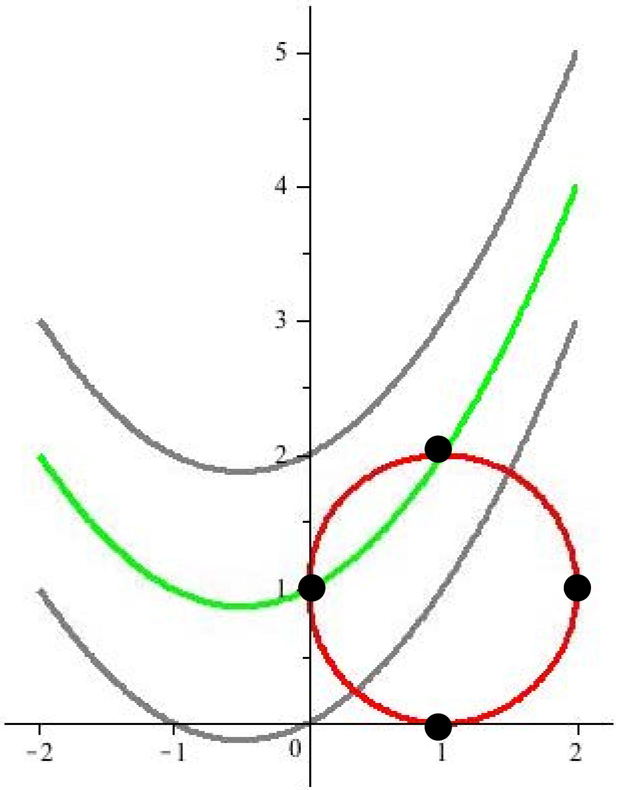
Example

$$\min \quad f(x) = x_1^2 + x_1 - 2x_2$$

$$s.t.:$$

$$x_1^2 + x_2^2 - 2x_1 - 2x_2 = -1$$

$$0 \le x_i \le 2 \quad \text{integers}$$



Optimal value: -2

Phase 1

SDP relaxation

$$\min \quad X_{11} + x_1 - 2x_2$$

$$s.t.:$$

$$X_{11} + X_{22} - 2x_1 - 2x_2 = -1$$

$$X_{ij} \le 2x_i$$

$$X_{ij} \ge 2x_i + 2x_j - 4 \, , X_{ii} \ge x_i ,$$

$$X_{ij} \ge 0$$

$$\begin{pmatrix} 1 & x^T \\ x & X \end{pmatrix} \succ 0$$

Optimal value: -2.5

$$S_0^* = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}$$

$$S_1^* = 0$$

$$\min \quad f(x) = x_1^2 + x_1 - 2x_2$$
$$s.t.:$$
$$x_1^2 + x_2^2 - 2x_1 - 2x_2 = -1$$
$$0 \le x_i \le 2 \quad \text{integers}$$

Phase 2

Reformulated problem

$$\min \quad f(x) = 2x_2^2 + x_1 - 2x_2 + y_{11} - 2y_{22}$$
$$s.t.:$$
$$y_{11} + y_{22} - 2x_1 - 2x_2 = -1$$
$$0 \le x_i \le 2 \quad \text{integer}$$
$$(x, y, z, t) \in P_{xyzt}$$

Optimal value: -2

Continuous relaxation value: -2.5

Comparison with linearization

$$\min \quad f(x) = x_1^2 + x_1 - 2x_2$$

$$s.t.:$$

$$x_1^2 + x_2^2 - 2x_1 - 2x_2 = -1$$

$$0 \le x_i \le 2 \quad \text{integers}$$

$$\min \quad f(x) = y_{11} + x_1 - 2x_2$$

$$s.t.:$$

$$y_{11} + y_{22} - 2x_1 - 2x_2 = -1$$

$$0 \le x_i \le 2 \quad \text{integer}$$

$$(x, y, z, t) \in P_{xyzt}$$

Optimal value: -2

Continuous relaxation value: -3

Numerical results

- 1 quadratic objective, 1 quadratic inequality
- u=20 - Averages on 10 instances/row
- B&B solution by Cplex 12.5 (limit 3 hours)

	(QP*)				(MILP)		
n	root gap	ref cpu	tot cpu	nodes	root gap	tot (s)	nodes
10	3.6%	1''	2.6''	134	47.5%	0.4''	169
20	2%	1'	1' 28''	475	88%	2' 34''	7487
30	1.5%	6' 20''	9'	739	124%	(0)	
40	2.5%	33'15''	1h (9)	4800	166%	(0)	

Conclusions

- General reformulation scheme for exact solution method

- Includig linearization

- An « optimal » reformulation is found from an SDP relaxation

- Captures the strengh of the SDP bound in the reformulation

- Extension to continuous non convex QP