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IBM ILOG CPLEX

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Algorithms and Solvers for Mixed Integer Nonlinear (Convex) Optimization

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Introduction

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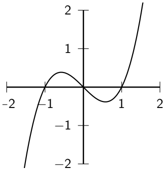
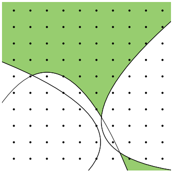
The mother of all deterministic optimization problems [Lee, 2008]

$$
\begin{array}{lll}
\min & f(x) & \\
\text{s.t.} & g_i(x) \le 0 & i = 1, \ldots, m \\
& x \in X & \\
& x_j \in \mathbb{Z} & j = 1, \ldots, p \\
& l_j \le x_j \le u_j & j = 1, \ldots, p
\end{array}
\qquad \text{(MINLP)}
$$

- $X \subseteq \mathbb{R}^n$ polyhedral.
- f and $g_i : X \to \mathbb{R}$, $i = 1, \ldots, m$, continuous, differentiable.





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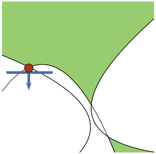
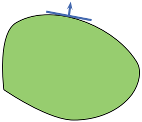
"Well solved" subproblems

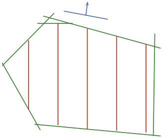
Nonlinear Programming (NLP)

$p = 0$: local optima. $\quad + f$ and g_i convex $\Rightarrow$ global optima.





Mixed-Integer linear programming (MILP)



- f linear, $m = 0$, $p > 0$

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The complexity issue

Theorem ([Jeroslow, 1973])

The problem of minimizing a linear form over quadratic constraints in integer variables is not computable by a recursive function.

Theorem ([De Loera et al., 2006])

The problem of minimizing a linear function over polynomial constraints in at most 10 integer variables is not computable by a recursive function.

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The complexity issue

There is no algorithm to solve (MINLP) ...

Theo

The p ... minimizing a linear form over quadratic constraints in integer variables is not computable by a recursive function.

Theorem ([De Loera et al., 2006])

The problem of minimizing a linear function over polynomial constraints in at most 10 integer variables is not computable by a recursive function.

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The complexity issue

There is no algorithm to solve (MINLP) ...

Theo...

The p... minimizing a linear form over quadratic constraints in integer variables is not computable by a recursive function.

... even with 10 variables.

Theorem ([De Loera ...

The ... over polynomial constraints in at most ... not computable by a recursive function.

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More complexity results

In any dimension

- Mixed Integer Quadratic Programming in in NP [Del Pia et al., 2015].

In fixed dimension

- There is an FPTAS for minimizing a non-negative polynomial over the intersection of a polyhedron and integers [De Loera et al., 2006].
- Cubic and homogeneous polynomials can be minimized over integers in the plane in polynomial time [Del Pia et al., 2014].
- Minimizing polynomials of degrees four in the plane is NP-hard [De Loera et al., 2006].

Many more results...

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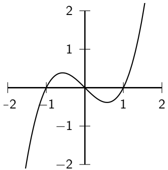
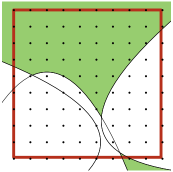
MINLP

$$
\begin{array}{lll}
\min & f(x) & \\
\text{s.t.} & g_i(x) \le 0 & i = 1, \dots, m \\
& x \in X & \qquad\qquad (\text{PNLM}) \\
& x_j \in \mathbb{Z} & j = 1, \dots, p \\
& {\color{red} l_j \le x_j \le u_j} & {\color{red} j = 1, \dots, p}
\end{array}
$$

- **To be solvable in general, l_j, u_j finite**.





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Two main classes of MINLP

Mixed Integer Convex Program

Assume that the continuous relaxation is a convex optimization problem.

- f is a convex function.
- g_i are convex functions.

Mixed Integer Nonlinear Program (or Global Optimization)

Don't assume any convexity on f or g_i.

- Continuous relaxation is NP-hard to solve in general.
- Remark: if l_j and u_j are finite integers, variable x_j can be seen as a continuous variable satisfying:

$$(x_j - l_j)(x_j - l_j - 1)....(x_j - u_j) = 0$$

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A special class of convex MINLP: MISOCP

$$
\begin{array}{lll}
\min & c^T x & \\
& x^T Q_k x + a_k^T x \leq a_k^0 & k = 1, \ldots, m, \\
& Ax = b, & \\
& x_j \in \mathbb{Z} & j = 1, \ldots, p.
\end{array}
\qquad \text{(MIQCP)}
$$

Where all quadratic constraints can be represented as second order cones (or Lorentz cone):

$$L^d := \{(x, x_0) \in \mathbb{R}^{d+1} : \sum_{i=1}^{d} x_i^2 \leq x_0^2, x_0 \geq 0\}.$$

(L^d defines the $(d+1)$-dimensional second order cone.)

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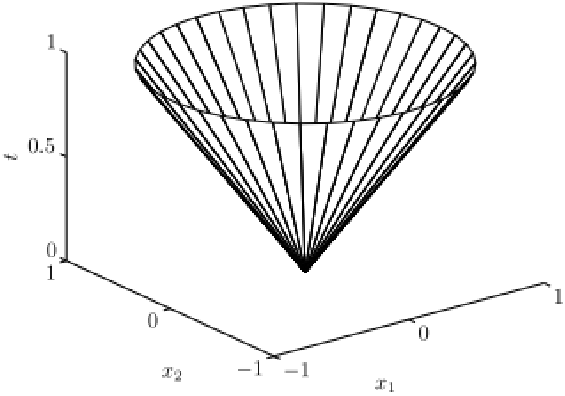
A Lorentz cone



It is convex!

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Second order cone representability

Through simple algebra can be represented as second order cones:

- Second order cones: $\sum_{i=1}^{d} x_i^2 \leq x_0^2$, with $x_0 \geq 0$
- Rotated second order cones: $\sum_{i=2}^{d} x_i^2 \leq x_0 x_1$, with $x_0, x_1 \geq 0$
- Simple convex quadratic constraints:

$$x^T Qx + a^T x \leq a^0, \text{ with } Q \succeq 0$$

- or more complicated...

$$||x^T Qx + a^T x|| \leq c^T x + b, \text{ with } Q \succeq 0$$

(the first three should be recognized by most solvers, the last one not.)

Many non-linear constraints can be formulated as second order cones but modeling may be very far from obvious.

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MISOCP

$$
\begin{aligned}
\min \quad & c^T x \\
& (x_{J_i}, x_{h_i}) \in L^{d_i} \quad && i = 1, \dots, m \\
& Ax = b, \\
& x_j \in \mathbb{Z} && j = 1, \dots, p.
\end{aligned}
\qquad \text{(MISOCP)}
$$

MINLP's where all nonlinear constraints are SOC

- Continuous relaxation solved efficiently by interior points.
- convex MINLP algorithms work with some added technicality due to non-differentiability [Drewes, 2009, Drewes and Ulbrich, 2012].
- Supported by most MIP solvers (all the ones you can think of).

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MISOCP Applications

Application	SOC	Integer
Portfolio optimization	Risk, utility, robustness	number of assets, min investment
[Bienstock, 1996, Bonami and Lejeune, 2009, Vielma et al., 2008]		
Truss topology optimization	Physical forces	Cross section of bars
[Achtziger and Stolpe, 2006]		
Networks with delays	Delay as function of traffic	Path, flows
[Boorstyn and Frank, 1977, Ameur and Ouorou, 2006]		
Location with stochastic services	Demands	location model
[Elhedhli, 2006]		
TSP with neighborhoods (Robotics)	Definition of ngbh.	TSP
[Gentilini et al., 2013]		
Many more... see for eg. http://cblib.zib.de.		

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Mixed Integer Convex Programming Applications (not MISOCP)

Application	nonlinear	discrete
Chemical plant design [Duran and Grossmann, 1986, Flores-Tlacuahuac and Biegler, 2007]	Chemical reactions	what to install
Block Layout Design [Castillo et al., 2005]	Spatial constraints	what to layout

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Mixed Integer Nonlinear Programming Applications

Application	nonlinear	discrete
Petrochemical [Haverly, 1978]	Blending, pooling	–
Power industry [Carpentier, 1962]	AC power flow	–
Gaz/Water networks	gaz/liquid flow equations	choice of pipes
[Koch et al., 2015, Bragalli et al., 2011]		
Nuclear Reactor reloading [Quist et al., 1999]	reactions	What to reload
Airplane trajectories	aerodynamics	waypoints, colisions
[Cafieri and Durand, 2013, Soler et al., 2013]		
Mixed Integer Optimal control [Sager, 2005, 2012]	DE	discrete controls
Countless more	. . .	. . .
see for example [Belotti et al., 2013b]		

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Agenda

- Part I: The Basic Algorithms for MICP.
  - Main Algorithmic Approaches
  - Glimpse of Computations
  - Glimpse of MISOCP
- Part II: Selected Advanced (or not) Topics.
  - Disaggregatin of nonlinear constraints
  - Lift-and-project cuts for MICP

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Part I

The Basic Algorithms

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The Basic Algorithms

Section 1

The convex case

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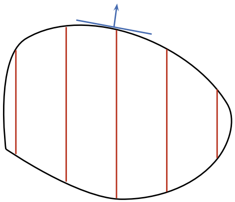
The mixed integer convex program

$$
\begin{array}{lll}
\min & c^T x & \\
\text{s.t.} & g_i(x) \leq 0 & i = 1, \ldots, m \\
& x \in X & \\
& x_j \in \mathbb{Z} & j = 1, \ldots, p
\end{array}
\qquad \text{(MICP)}
$$



- $g_i : X \to \mathbb{R}$, $i = 1, \ldots, m$, *convex, differentiable.*
- Assume linear objective. If necessary, add var $\alpha \in \mathbb{R}$ and $\min \alpha$ with $f(x) \leq \alpha$ a constraint.

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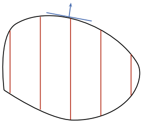
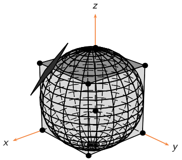
Main Algorithms for solving (MICP)





Fundamental property is convexity of the continuous relaxation, which can be efficiently solved.

1. NLP Branch-and-bound [Gupta and Ravindran, 1985].
2. Outer Approximation Algorithm [Duran and Grossmann, 1986]. Builds an MILP equivalent of (MICP)
3. LP/NLP branch-and-cut [Quesada and Grossmann, 1992].

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NLP based branch-and-bound

Straightforward generalization of main MILP algorithm:

- solve an NLP at each node of the tree.

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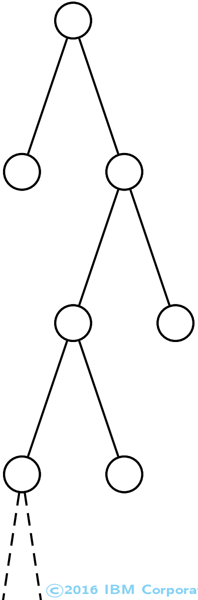
NLP based branch-and-bound



Straightforward generalization of main MILP algorithm:

- solve an NLP at each node of the tree.
- Branch on variables with fractional value.

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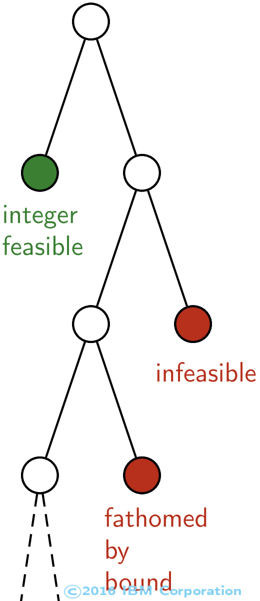
NLP based branch-and-bound

Straightforward generalization of main MILP algorithm:

- solve an NLP at each node of the tree.
- Branch on variables with fractional value.
- Prune by infeasibility, bounds and integer feasibility.



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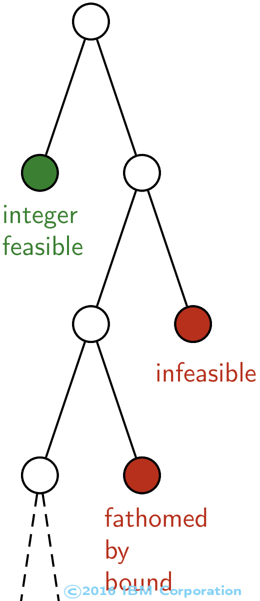
NLP based branch-and-bound

Straightforward generalization of main MILP algorithm:

- solve an NLP at each node of the tree.
- Branch on variables with fractional value.
- Prune by infeasibility, bounds and integer feasibility.

Main issues

- Warm-starting of NLP solves.
- Stability of NLP solvers.
- Difficulty of reusing MILP technologies.



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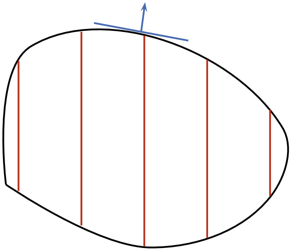
Outer Approximation [Duran and Grossmann, 1986]



$$
\begin{aligned}
&\min \quad c^T x \\
&\text{s.t.} \\
&g_i(x) \leq 0 \qquad i = 1, \ldots, m, \\
&x_j \in \mathbb{Z}, \qquad j = 1, \ldots, p.
\end{aligned}
$$

Idea: Take first-order approximations of constraints at different points and build an equivalent MILP.

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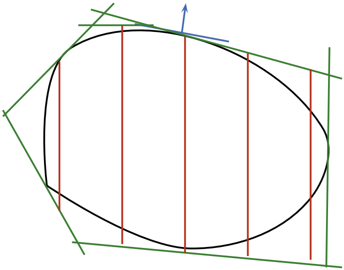
Outer Approximation [Duran and Grossmann, 1986]



$$\min \quad c^T x$$

s.t.

$$g_i(x) \leq 0 \qquad i = 1, \ldots, m,$$

$$x_j \in \mathbb{Z}, \qquad j = 1, \ldots, p.$$

Idea: Take first-order approximations of constraints at different points and build an equivalent MILP.

$$\min \quad c^T x$$

s.t.

$$g_i(\overline{x}^k) + \nabla g_i(\overline{x}^k)^T (x - \overline{x}^k) \leq 0 \qquad i = 1, \ldots, m, \; k = 1, \ldots, K$$

$$x_j \in \mathbb{Z}, \qquad j = 1, \ldots, p.$$

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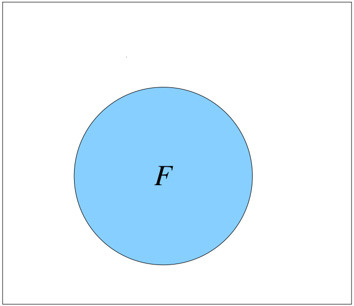
Outer approximation constraints

Let $F := \{x : x \in X : g_i(x) \leq 0\}$
$(g_i : \mathbb{R}^n \to \mathbb{R}$ convex. $)$
Outer approximation constraint in $\bar{x}$:

$$\nabla g_j(\bar{x})^T (x - \bar{x}) + g_j(\bar{x}) \leq g_j(x) \leq 0.$$

(valid for F by convexity of g_j and definition of F.)



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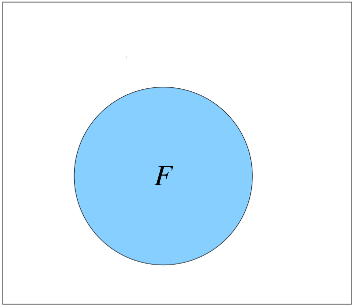

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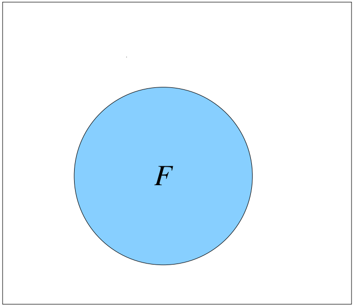

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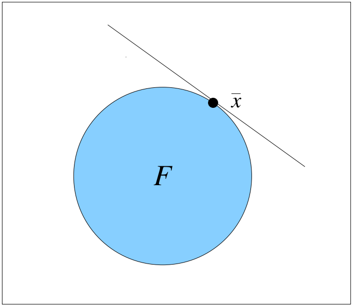
(valid for F by convexity of g_j and definition of F.)

- If $g(\bar{x}) = 0$ tangent to feasible region.
- If $g(\bar{x}) < 0$ non-tight constraint.
- If $g(\bar{x}) > 0$ non-tight constraint cutting off $\bar{x}$.



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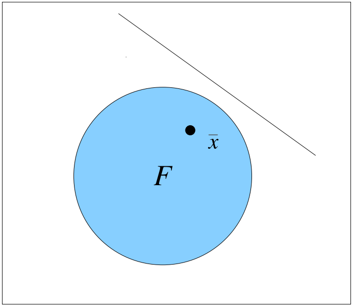

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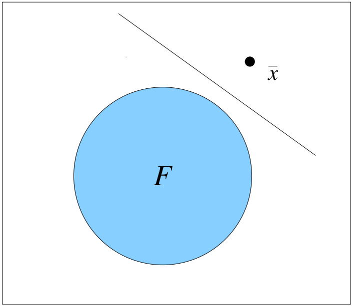

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Subproblems

Given $\hat{x} \in \mathbb{R}^p$:

fixed NLP (NLP($\hat{x}$))

$$\begin{aligned} &\min\ c^T x \\ &s.t. \\ &g_i(x) \leq 0, && i = 1, \ldots, m \\ &x \in X; && (\text{NLP}(\hat{x})) \\ &x_j = \hat{x}_j, && j = 1, \ldots, p. \end{aligned}$$

If $\hat{x} \in \mathbb{Z}^p$, and feasible: gives an upper bound.

fixed feasibility subproblem

$$\begin{aligned} &\min \sum_{i=1}^{m} w_i \max\{0, g_i(x)\} \\ &\text{s.t.} \\ &x \in X, && (\text{NLPF}(\hat{x})) \\ &x_j = \hat{x}_j,\ j = 1, \ldots, p \end{aligned}$$

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Subproblems

Given $\hat{x} \in \mathbb{R}^p$:

fixed NLP (NLP($\hat{x}$))

$$
\begin{aligned}
&\min\ c^T x \\
&s.t. \\
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\end{aligned}
$$

If $\hat{x} \in \mathbb{Z}^p$, and feasible: gives an upper bound.

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$$
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&\text{s.t.} \\
&x \in X, \qquad\qquad (\text{NLPF}(\hat{x})) \\
&x_j = \hat{x}_j,\ j = 1, \ldots, p
\end{aligned}
$$

Remark If (NLP($\hat{x}$)) is infeasible, NLP software will typically return a solution to (NLPF($\hat{x}$)). By abuse, always say solution to (NLP($\hat{x}$))

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Equivalent MILP formulation of convex MINLP

For each $\hat{x}^k \in K = \mathrm{Proj}_{1,\ldots,p}(X) \cap \mathbb{Z}^p$, let $\overline{x}^k$ be an optimal solution to (NLP($\hat{x}$)).

Theorem ([Duran and Grossmann, 1986])

If $X \neq \emptyset$, f and g are convex, continuously differentiable, and a constraint qualification holds for each $\overline{x}^k$ then

$$\begin{aligned} &\min \quad c^T x \\ &g_i(\overline{x}^k) + \nabla g_i(\overline{x}^k)^T (x - \overline{x}^k) \leq 0 \quad i = 1, \ldots, m, \hat{x}^k \in K, \\ &x \in X,\ x_j \in \mathbb{Z},\ j = 1, \ldots, p. \end{aligned}$$

has the same optimal value as (MICP).

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OA decomposition

Generate MILP equivalent by constraint generation.

- Initialize with one set of linearizations.

$$\begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad & \\ & g_i(\overline{x}^0) + \nabla g_i(\overline{x}^0)^T (x - \overline{x}^0) \leq 0, \qquad i = 1, \ldots, m, \quad , \qquad (\mathrm{OA}(\mathcal{K})) \\ & x \in X,\ x_j \in \mathbb{Z},\ j = 1, \ldots, p. \end{aligned}$$

Where x^0 is the solution to the continuous relaxation:
$\min\{c^T x : x \in X,\ g_i(x) \leq 0, i = 1, \ldots, m\}$

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OA decomposition

Generate MILP equivalent by constraint generation.

- Initialize with one set of linearizations.
- Enrich iteratively the set of linearizations $\mathcal{K}$.

$$\begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad & \\ & g_i(\overline{x}^k) + \nabla g_i(\overline{x}^k)^T (x - \overline{x}^k) \leq 0, \qquad \begin{array}{c} i = 1, \ldots, m, \\ \hat{x}^k \in \mathcal{K} \end{array}, \\ & x \in X,\ x_j \in \mathbb{Z},\ j = 1, \ldots, p. \end{aligned} \tag{OA($\mathcal{K}$)}$$

Where $\hat{x}^k$ is a solution to (OA($\mathcal{K}$)) and, for $k = 1, \ldots, |\mathcal{K}|$, $\overline{x}^k$ is the solution to (NLP($\hat{x}$)).

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OA decomposition

Generate MILP equivalent by constraint generation.

- Initialize with one set of linearizations.
- Enrich iteratively the set of linearizations $\mathcal{K}$.

Convergence

At each iteration:

- (OA($\mathcal{K}$)) gives a lower bound,
- If feasible, (NLP($\hat{x}$)) gives an upper bound.

The OA Theorem guarantees that the two bounds converge in finite # of iterations.

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Outer-Approximation Decomposition Algorithm

0. **Initialize.**
   $z_U \leftarrow +\infty$. $z_L \leftarrow -\infty$. Let $\overline{x}^0$ be the optimal solution of continuous relaxation.
   $\mathcal{K} \leftarrow \left\{\overline{x}^0\right\}$. Choose a convergence tolerance ϵ.
1. **Terminate?**
   Is $z_U - z_L < \epsilon$ or $(\mathrm{OA}(\mathcal{K}))$ infeasible? If so z_U is $\epsilon-$optimal.
2. **Lower Bound**
   Let $z_{\mathrm{MP}(\mathcal{K})}$ be the optimal value of $\mathrm{OA}(\mathcal{K})$ and $(\hat{x})$ its optimal solution.
   $z_L \leftarrow z_{\mathrm{MP}(\mathcal{K})}$
3. **NLP Solve**
   Solve $(\mathrm{NLP}(\hat{x}))$.
   Let $\overline{x}^i$ be the optimal (or minimally infeasible) solution.
4. **Upper Bound?**
   Is $\overline{x}^i$ feasible for (MINLP)? If so, $z_U \leftarrow \min(z_U, f(\overline{x}^i))$.
5. **Refine**
   $\mathcal{K} \leftarrow \mathcal{K} \cup \{\overline{x}^i\}$ and $i \leftarrow i + 1$.
   Go to 1.

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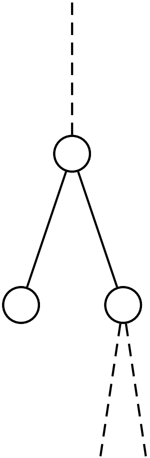
LP/NLP Branch-and-bound

OA can be embedded in a single tree search.

- Start solving the same initial MILP by branch-and-bound.
- At each integer feasible node:



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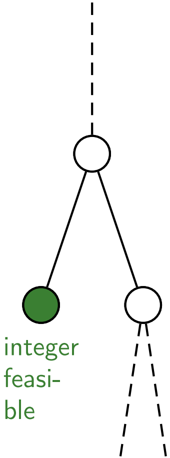
LP/NLP Branch-and-bound

OA can be embedded in a single tree search.

- Start solving the same initial MILP by branch-and-bound.
- At each integer feasible node:
  1. solve (NLP($\hat{x}$)), and enrich the set of linearizations.
  2. Resolve the LP relaxation of the node with the new cuts.
  3. Repeat as long as node is integer feasible.



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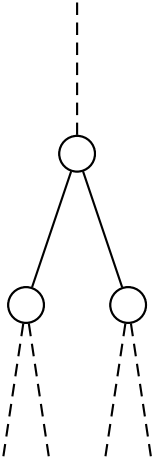
LP/NLP Branch-and-bound

OA can be embedded in a single tree search.



- Start solving the same initial MILP by branch-and-bound.
- At each integer feasible node:
  1. solve (NLP($\hat{x}$)), and enrich the set of linearizations.
  2. Resolve the LP relaxation of the node with the new cuts.
  3. Repeat as long as node is integer feasible.
- Never prune by integer feasibility.

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Solvers for Mixed Integer Convex Programs

Solver	Reference	Algorithm(s)
Dicopt		OA
MINLP_BB	[Leyffer, 1998]	NLP BB
SBB	[Bussieck and Drud, 2001]	NLP BB
α-ECP	[Westerlund and Lundqvist, 2005]	ECP (variant of OA)
Bonmin	[Bonami et al., 2008]	NLP BB, OA, LP/NLP
FilMINT	[Abhishek et al., 2010]	LP/NLP
KNITRO	[Byrd et al., 2006]	NLP BB, LP/NLP
SCIP	[Vigerske, 2012]	LP/NLP

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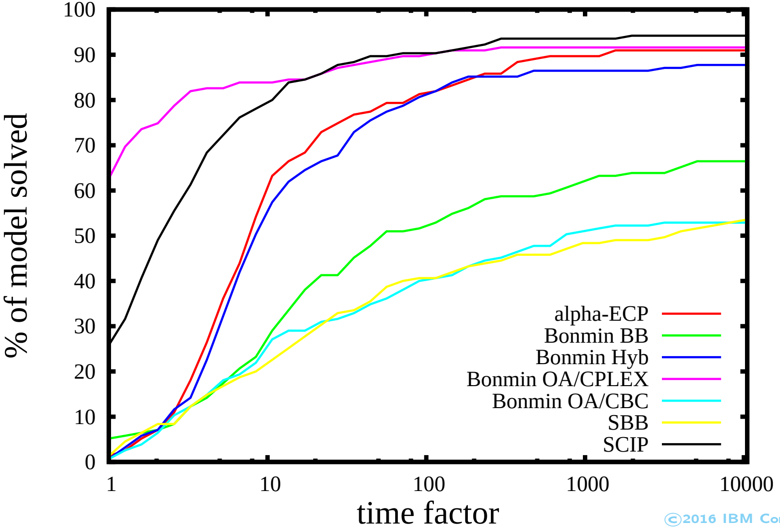
Comparison of solvers in GAMS [Vigerske, 2013]



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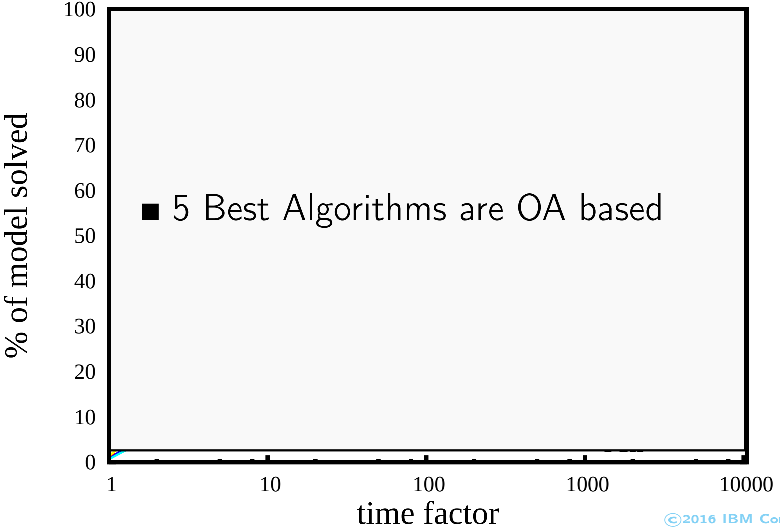
Comparison of solvers in GAMS [Vigerske, 2013]



- 5 Best Algorithms are OA based

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Notes on results with Bonmin

- Bonmin's OA using CPLEX seems the best algorithm overall.
  - It is also the simplest: a loop calling CPLEX (MILP) and Ipopt (NLP) alternatively as black boxes.
  - Improves with CPLEX.
- Bonmin's Hyb is in the pack of relatively good solvers
  - own variant of LP/NLP BB.
  - Reuse CBC infrastructure, LP solver, Cuts, MIP presolve.
  - Improves at a slower pace.
- Bonmin's BB clearly behind.
  - pure NLP based branch-and-bound. Doesn't reuse much from Cbc. Everything specifically tailored.
  - Better implementation exists that should be on par with Hyb.
- Bonmin's OA using CBC seems the worst algorithm overall.

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The MIQCP/MISOCP solver in CPLEX

Implements the two main algorithms:

- A branch-and-bound based on the continuous SOCP solver (barrier).
- An outer approximation branch-and-cut algorithm.

Choice is controled by the parameter `CPXPARAM_MIP_Strategy_MIQCPStrat`.
Default is trying to decide which of the two algorithms to run in a "clever" way.

History of MIQCP with CPLEX

class	p	algorithm	V. (Year)
Convex QCP	0	barrier	9.0 (2003)
convex MIQCP	> 0	barrier based B&B	9.0 (2003)
–	–	Outer approximation B&C	11.0 (2007)

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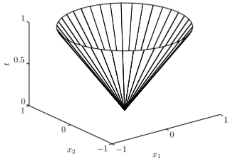
MISOCP specifics

Remember our assumption for MICP algorithms and the Lorentz cone:

Assumption:
$g_i : X \to \mathbb{R}$, $i = 1, \ldots, m$, convex, differentiable.
We have:



$$g_i(x) = \sum_{i=1}^{d} x_i^2 - x_0^2, \text{ with } x_0 \geq 0$$

The assumption is not verified at the head of the cone (when $x_0 = 0$), in particular Outer Approximation can't be applied as is.

In practice

[Drewes, 2009, Drewes and Ulbrich, 2012] recommend a hybrid OA approach where the SOCP relaxation is applied and used at integer feasible nodes where we are at the head of a cone.

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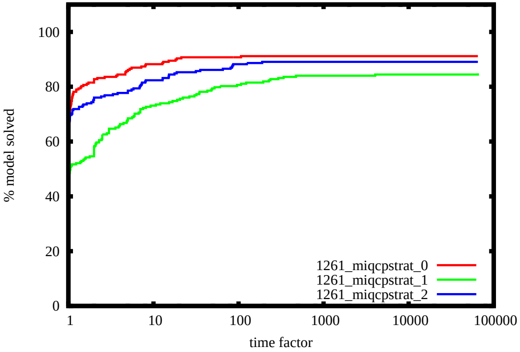
A comparison of OA and SOCP-BB in CPLEX 12.6.1 [1]



Default strategy picked

- OA 113 times
- SOCP-BB 46 times
- 56 models identical with both

To be perfect should have picked

- 14 more models with OA
- 36 more models with SOCP-BB

[1] 225 models solved by at least one method and failed by none.

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Advanced algorithms for convex case (non exhaustive references)

- Preprocessing/Modeling:
  - separability [Hijazi et al., 14]
  - perspective formulations [Frangioni and Gentile, 2006, Günlük and Linderoth, 2008]
  - propagation [Vigerske, 2012]
- Node relaxations/Branching:
  - QP Delaxations in strong-branching [Bonami et al., 2013]
  - QP Divings [Mahajan et al., 2012]
- Primal Heuristics:
  - Feasibility Pumps [Bonami et al., 2009],
  - Undercover [Berthold and Gleixner, 2013]
- Cuts:
  - Disjunctive Cuts [Kılınc et al., 2011, Bonami, 2011].
  - Conic Cuts for Conic Crogramming [Andersen and Jensen, 2013, Belotti et al., 2013a, Kılınç-Karzan and Yıldız, 2015, Modaresi et al., 2015] (among others)

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Part II

Selected Advanced (or not) Topics

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Selected Advanced (or not) Topics

Section 2

Disaggregation of nonlinear constraint

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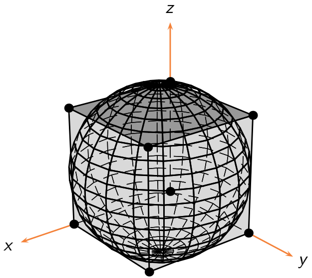
A most simple MINLP

Consider the following convex MINLP:

$$
\begin{aligned}
\min \quad & \sum_{i=1}^{n} i \times x_i \\
\text{s.t.} \quad & \sum_{i=1}^{n} \left(x_i - \tfrac{1}{2}\right)^2 \leq \tfrac{n-1}{4} \qquad (1) \\
& x \in \mathbb{Z}^n
\end{aligned}
$$



Exercise

- Find the optimum or prove that (1) is infeasible or unbounded.
- How many nodes, would a simple branch-and-bound take to solve (1)?
- How many linear approximations would an Outer Approximation approach need?

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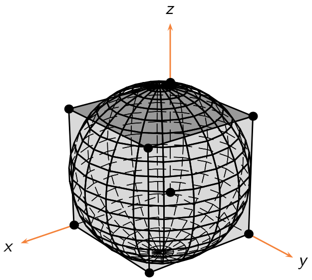
Answers

Consider the following convex MINLP:

$$
\begin{array}{ll}
\min & \sum_{i=1}^{n} i \times x_i \\
\text{s.t.} & \sum_{i=1}^{n} \left(x_i - \frac{1}{2}\right)^2 \leq \frac{n-1}{4} \\
& -10 \leq x \leq 10, x \in \mathbb{Z}^n
\end{array}
\qquad (1)
$$



- (1) is infeasible:
  - The ball is too small to contain integer points.
  - It is large enough to touch every edge of the hypercube.
- A basic branch-and-bound would take at least 2^{n+1} nodes.
- We need at least 2^n linear outer approximations to prove infeasibility.

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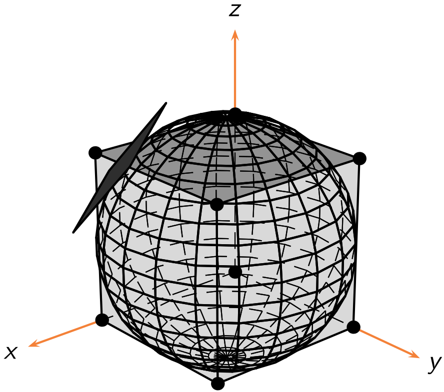
Solving (1) with OA cuts

- No OA constraint can cut 2 vertices of the hypercube.
  - If an inequality cuts two points, it cuts the segment joining them.
  - The ball has a non-empty intersection with every segment joining two vertices.
  - Remember that an outer approximation is only a tangent to the ball.



What do solvers tell?

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Solving (1) with OA cuts

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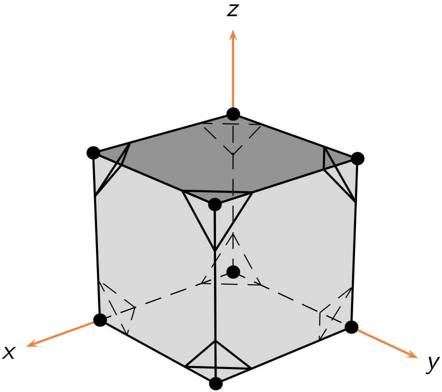

What do solvers tell?

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Four years ago

		CPLEX 12.4	SCIP 2.0.1	B-OA	B-Hyb
n	2^n	nodes	nodes	OA it.	nodes

Four years ago

		CPLEX 12.4	SCIP 2.0.1	B-OA	B-Hyb
n	2^n	nodes	nodes	OA it.	nodes
10	1,024	2,047	720	1,105	11,156

Four years ago

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Remark

- Problem is trivial if variables are $0-1$: replace x_i^2 by x_i, the contradiction $\frac{n}{4} \leq \frac{n-1}{4}$ follows.

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Four years ago

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Remark

- Problem is trivial if variables are $0-1$: replace x_i^2 by x_i, the contradiction $\frac{n}{4} \leq \frac{n-1}{4}$ follows.
- SCIP $\geq$ 2.1 and CPLEX $\geq$ 12.6.1 solve it in a blink.

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Solving the problem by presolve/propagation

An easy way to deduce infeasibility is to compute the component-wise maximum of the left-hand-side of the constraint:

$$\sum_{i=1}^{n} \min \left\{ \left(x_i - \frac{1}{2} \right)^2 : x_i \in \mathbb{Z} \right\}$$

Each optimization problem is one dimensional and can be easily solved:

$$\min \left\{ \left(x_i - \frac{1}{2} \right)^2 : x_i \in \mathbb{Z} \right\} = \frac{1}{4}$$

Summing up we get that:

$$\sum_{i=1}^{n} \min \left\{ \left(x_i - \frac{1}{2} \right)^2 : x_i \in \mathbb{Z} \right\} = \frac{n}{4} > \frac{n-1}{4}.$$

A contradiction, therefore (1) is infeasible.

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Twisting our example

The following model should be complicated enough to pass presolve untouched:

$$\begin{aligned} &\min \sum_{i=1}^{2n} i * x_i \\ &\sum_{i=1}^{n} (100x_{2i}^2 + 100x_{2i-1}^2 - 4x_{2i}x_{2i-1} - 98x_{2i} - 98x_{2i-1}) \leq -1 \\ &-10 \leq x \leq 10, x \in \mathbb{Z}^{2n} \end{aligned} \tag{2}$$

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A recipe for solving (2) better with OA

We consider a specific class of MINLPs:

$$\begin{array}{lll} \min & c^T x & \\ \text{s.t.} & g_i(x) \leq 0 & i = 1, \dots, m \\ & x \in X & \\ & x_j \in \mathbb{Z} & j = 1, \dots, p \\ & l \leq x \leq u & \end{array} \qquad \text{(sMINLP)}$$

- For $i = 1, \dots, m$, $g_i : X \to R$ are *convex separable*:

$$g_i(x) = \sum_{j=1}^{n} g_{ij}(x_j)$$

with $g_{ij} : [l_j, u_j] \to \mathbb{R}$ convex.

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Disaggregated formulation

Introduce one variable y_{ij} for each elementary function:

$$
\begin{aligned}
\min \quad & c^T x \\
\text{s.t.} \quad & \sum_{j=1}^{n} y_{ij} \leq 0 && i = 1, \dots, m, \\
& g_{ij}(x_j) \leq y_{ij} && \begin{array}{l} i = 1, \dots, m, \\ j = 1, \dots, n, \end{array} \\
& x \in X, \\
& x_i \in \mathbb{Z} && i = 1, \dots, p, \\
& l \leq x \leq u.
\end{aligned}
\qquad \text{(sMINLP}^*\text{)}
$$

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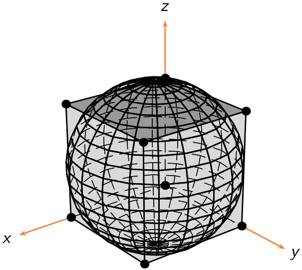
Application to (1)

Extended formulation of (1)

$$\begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad & \sum_{i=1}^{n} y_i \le (n-1)/4 \\ & (x_i - 0.5)^2 \le y_i \qquad i = 1, \dots, n \\ & x \in \mathbb{Z}^n. \end{aligned} \tag{3}$$



Its outer approximation

$$\begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad & \sum_{i=1}^{n} y_i \le (n-1)/4 \\ & 2\left(\overline{x}_i^k - 0.5\right)\left(x_i - \overline{x}_i^k\right) + \left(\overline{x}_i^k - 0.5\right)^2 \le y_i \qquad \begin{matrix} i = 1, \dots, n \\ k = 1, \dots, K \end{matrix} \\ & x \in \mathbb{Z}^n \end{aligned}$$

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Application to (1)

Extended formulation of (1)

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\begin{aligned}
\min\ & c^T x \\
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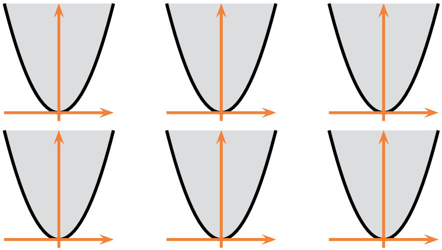
$$



Its outer approximation

$$
\begin{aligned}
\min\ & c^T x \\
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& 2\left(\bar{x}_i^k - 0.5\right)\left(x_i - \bar{x}_i^k\right) + \left(\bar{x}_i^k - 0.5\right)^2 \le y_i \qquad \begin{matrix} i = 1, \ldots, n \\ k = 1, \ldots, K \end{matrix} \\
& x \in \mathbb{Z}^n
\end{aligned}
$$

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Application to (1)

Extended formulation of (1)

$$\min \; c^T x$$

$$\text{s.t.} \sum_{i=1}^{n} y$$

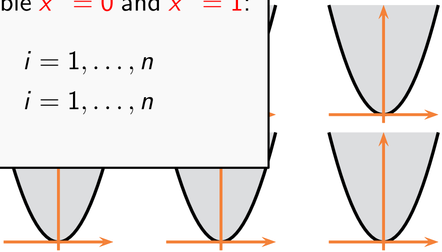
$$(x_i - 0.5) \le y_i \qquad i = 1, \dots, n$$

$$x \in \mathbb{Z}^n. \qquad (3)$$

2 points suffice to make it infeasible $\overline{x}^1 = 0$ and $\overline{x}^2 = 1$:

$$-x_i + 0.25 \le y_i \qquad i = 1, \dots, n$$

$$x_i - 0.75 + \le y_i \qquad i = 1, \dots, n$$



Its outer approximation

$$\min \; c^T x$$

$$\text{s.t.} \sum_{i=1}^{n} y_i \le (n-1)/4$$

$$2\left(\overline{x}_i^k - 0.5\right)\left(x_i - \overline{x}_i^k\right) + \left(\overline{x}_i^k - 0.5\right)^2 \le y_i \qquad \begin{matrix} i = 1, \dots, n \\ k = 1, \dots, K \end{matrix}$$

$$x \in \mathbb{Z}^n$$

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Application to (2): Solution

We need to get from this

$$\sum_{i=1}^{n}(100x_{2i}^2 + 100x_{2i-1}^2 - 4x_{2i}x_{2i-1} - 98x_{2i} - 98x_{2i-1}) \leq -1$$

to something of the form:

$$\begin{aligned} &\textstyle\sum_{i=1}^{n}(\alpha z_{2i} + \beta z_{2i-1} - 98x_{2i} - 98x_{2i-1}) \leq -1 \\ &y_i^2 \leq z_i \\ &y_i = \gamma_i^T x. \end{aligned}$$

How do we find $\alpha, \beta and \gamma$?
Spectral decompostion:

$$\begin{pmatrix} 100 & -2 \\ -2 & 100 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} 51 & 0 \\ 0 & 49 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ -1 & -1 \end{pmatrix}$$

$$\alpha = 51, \beta = 49, \gamma_{2i} = (-1, 1), \gamma_{2i-1} = (-1, -1)$$

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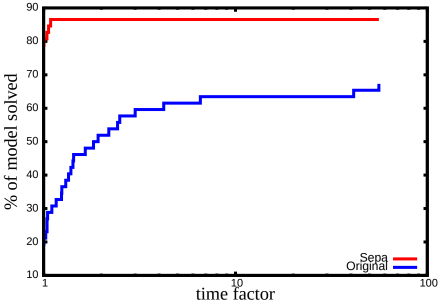
Experimental Illustration

- In the standard benchmark for MICP, out of $>$ 100 instances, 8 are not directly separable.
- Constructing separated formulations on a subset of 47 instances gives a 3x speed up: [Hijazi et al., 14].



Similar technique developped in Baron for compositions of convex functions [Tawarmalani and Sahinidis, 2004].

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Disaggregation of Second Order cones [Vielma et al., 2015]

In standard form the nonlinear constraint describing the second order cone is not convex separable:

$$\sum_{i=1}^{n} x_i^2 \leq x_0^2$$

Trick , divide the constraint by $x_0 \geq 0$ to get a convex separable constraint:

$$\sum_{i=1}^{n} \frac{x_i^2}{x_0} \leq x_0.$$

Now introduce $y_1, \ldots, y_n$ and rewrite as:

$$\sum_{i=1}^{n} y_i \leq x_0$$

$$x_i^2 \leq x_0 y_i$$

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General cones and Disciplined Convex Programming [Grant et al., 2006]

More general convex cones can be used to model convex problems:

$$EXP := cl\{(x, y, z) \in R^3 : ye^{\frac{x}{y}} \leq z, y \geq 0\},$$
$$POW_\alpha := \{(x, y, z) \in \mathbb{R}^3 : |z| \leq x^\alpha y^{1-\alpha}, x, y \geq 0\}$$

Adding these to MISOCP, convex problem can be modeled in a *disciplined* way by using composition of functions such that convexity is verifiable leading to DCP (see CVX [Grant and Boyd, 2014]) and MIDCP.
Out of 333 convex models in MINLPLIB2 [Vigerske, 2015] 333 can be reformulated in a disciplined manner [Lubin et al., 2015]!

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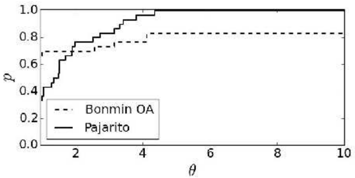
Disaggregation and MIDCP [Lubin et al., 2015]

The compositions used in DCP have a 1-1 correspondence with the existence of extended formulation disaggregating the cones and generalize the notion of separability.

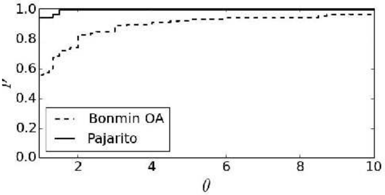
Pajarito sovler

- Solver for MIDCP implemented in Julia/JUMP [?Dunning et al., 2015].
- Claimed less than 1000 lines of codes.
- Dominates Bonmin OA on MINLPLIB2

Solution Time



Iterations



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Selected Advanced (or not) Topics

Section 3

Lift-and-project cuts for MICP

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Lift-and-project cuts for MISOCP

- Cuts are an essential component of MILP solvers
- Can always apply MILP cuts to a linear OA (and most people do)
- Can we generate better cuts by looking directly at nonlinear functions?
- A partial answer: as long as the cut generated is linear it could also have been obtained from a linear outer approximation
- In the past three years, tremendous activity towards conic cuts for conic programming [Andersen and Jensen, 2013, Belotti et al., 2013a, Kılınç-Karzan and Yıldız, 2015, Modaresi et al., 2015] (among others)

Our goal

- Derive linear cutting planes
- Fast
- Find an appropriate OA from which to derive a cut

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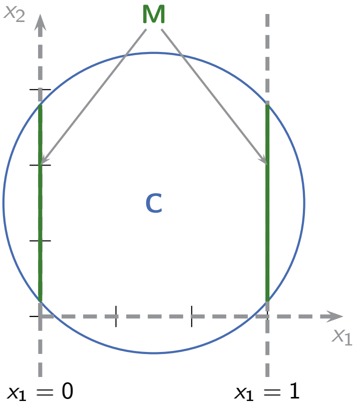
Split Relaxation

Consider $\mathbf{C}$ and $\mathbf{M} := \mathbf{C} \cap (\mathbb{Z}^p \times \mathbb{R}^{n-p})$.
Let $\pi \in \mathbb{Z}^p \times \{0\}^{n-p}$, $\pi_0 \in \mathbb{Z}$ and

$$\mathbf{C}^{(\pi,\pi_0)} := \text{conv}\Big(\mathbf{C} \cap \big(\{x : \pi^T x \le \pi_0\} \cup \{x : \pi^T x \ge \pi_0 + 1\}\big)\Big).$$

(clearly $\mathbf{M} \subseteq \mathbf{C}^{(\pi,\pi_0)} \subseteq \mathbf{C}$).



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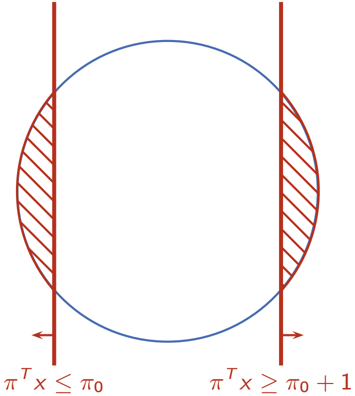
Split Relaxation

Consider **C** and **M** $:= \mathbf{C} \cap (\mathbb{Z}^p \times \mathbb{R}^{n-p})$.
Let $\pi \in \mathbb{Z}^p \times \{0\}^{n-p}$, $\pi_0 \in \mathbb{Z}$ and

$$\mathbf{C}^{(\pi,\pi_0)} := \mathrm{conv}\Big(\mathbf{C} \cap \big(\{x : \pi^T x \leq \pi_0\} \cup \{x : \pi^T x \geq \pi_0 + 1\}\big)\Big).$$

(clearly $\mathbf{M} \subseteq \mathbf{C}^{(\pi,\pi_0)} \subseteq \mathbf{C}$).



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Split Relaxation

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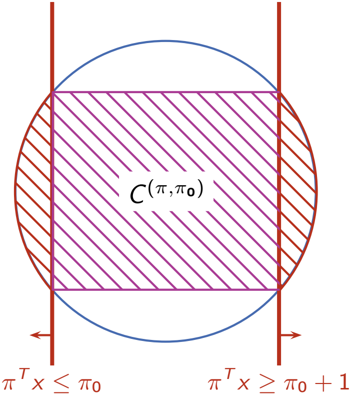

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Split Relaxation

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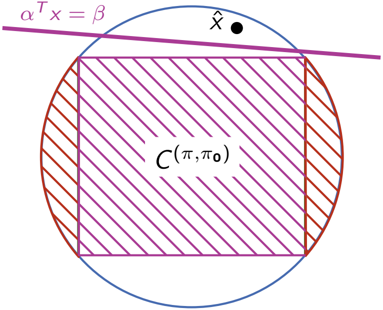
(clearly $\mathbf{M} \subseteq \mathbf{C}^{(\pi,\pi_0)} \subseteq \mathbf{C}$).

In the remainder, $\hat{x}$ is the point to separate,
$\pi = e_k$, $\hat{x}_k \in]0,1[$ $(k \leq p)$, and $\pi_0 = 0$



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MILP case

Consider a polyhedron $P := \{x : Ax = b, x \geq 0\}$

Cut Generation LP [Balas, 1979, Balas et al., 1993]

$\hat{x} \in P$ is separated from $P^{(e_k,0)}$ using the LP:

$$
\begin{aligned}
&\min \alpha^T \hat{x} - \beta \\
&s.t. : \\
&\alpha = u^T A - u_0 e_k, \qquad \alpha = v^T A + v_0 e_k, \\
&\beta = u^T b, \qquad \beta = v^T b + v_0, \\
&\alpha \in \mathbb{R}^n, \beta \in \mathbb{R},\ u, v \in \mathbb{R}^m,\ u_0, v_0 \in \mathbb{R}_+
\end{aligned} \tag{CGLP}
$$

If $\hat{x} \notin P^{(e_k,0)}$, $\alpha^T x \geq \beta$ cuts $\hat{x}$; otherwise produces certificate that $\hat{x} \in P^{(e_k,0)}$ with $x^0 \in P \cap \{x_k = 0\}$, $x^1 \in P \cap \{x_k = 1\}$ such that $\hat{x} = \hat{x}_k x^1 + (1 - \hat{x}_k) x^0$.

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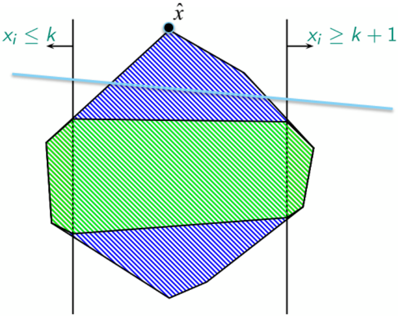
Statement in picture

If $\hat{x}$ is not in the split relaxation we get a cut.

$$\alpha = u^T A - u_0 e_k, \quad \beta = u^T b.$$



$$\alpha = v^T A + v_0 e_k, \quad \beta = v^T b + v_0.$$

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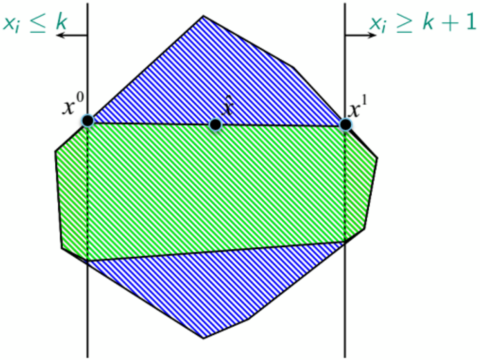
Statement in picture

If $\hat{x}$ is in the split relaxation we get a certificate in the form of two points x^0, x^1.



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MILP case (primal view)

Membership LP [Bonami, 2012]

$\hat{x} \in P$ wih $0 < \hat{x}_k < 1$ also in $P^{(e_k,0)}$ if $\exists\ x^0 \in P \cap \{x_k = 0\}$ and $x^1 \in P \cap \{x_k = 1\}$ with $\hat{x} = \hat{x}_k x^1 + (1 - \hat{x}_k)x^0$, or if

$$\begin{aligned} \max\ & y_k \\ \text{s.t.} & \\ & Ay = b\hat{x}_k \\ & 0 \leq y \leq \hat{x}, \\ & y \in \mathbb{R}^n. \end{aligned} \tag{MLP}$$

has a solution with $y_k = \hat{x}_k$ otherwise can deduce a cut from dual optimal solution.
(Hint $\frac{y}{\hat{x}_k}$ is x^1, $\frac{\hat{x}-y}{1-\hat{x}_k}$ is x^0).

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Generalization to MICPs

Using the primal view

- Generalizing (MLP) to nonlinear convex constraints is relatively simple [Bonami, 2011].
- But Nonlinear programming duality is not the same as LP!

Using the dual view

- Generalizing CGLP is possible but poses many numerical/technical challenges[Ceria and Soares, 1999, Stubbs and Mehrotra, 1999].
- As long as we generate a linear cut, it can be obtained from linear outer approximations[Bonami et al., 2012].
- The linear case can be used within a cut generation framework [Kılınc et al., 2011].

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Generalization to MICP: Two competing approaches

Goal: build a linear OA from which a "best" cut can be deduced using CGLP.

Using only LP [Kılınc et al., 2011].

1. Start with any linear OA of C
2. Solve CGLP. If a cut is found STOP otherwise
3. Deduce from dual of CGLP two points such that $\hat{x} = \lambda x^1 + (1-\lambda)x^0$ and satisfying the disjunction.
4. If point(s) not in C generate new OA and goto 2, otherwise there is no cut, STOP.

Using NLP [Bonami, 2011]

1. Solve a single NMLP that tells if $\hat{x}$ is in the split relaxation.
2. If not, deduce from solution two points such that $\hat{x} = \lambda x^1 + (1-\lambda)x^0$ and closest to the disjunction.
3. Build OA around these two points.
4. Solve MLP and get the cut.

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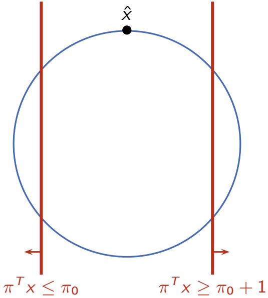
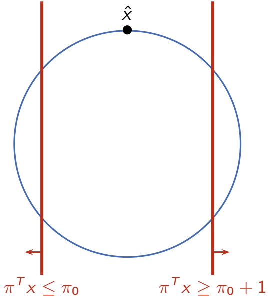
Generalization: Two competing approaches in pictures

Goal: build a linear OA from which a "best" cut can be deduced using CGLP.





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Generalization: Two competing approaches in pictures

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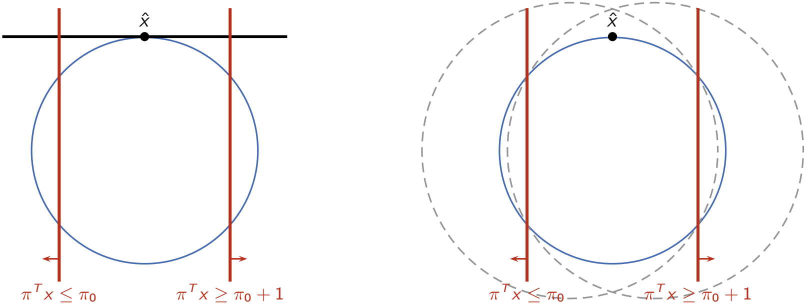

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Generalization: Two competing approaches in pictures

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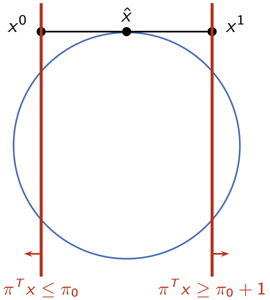

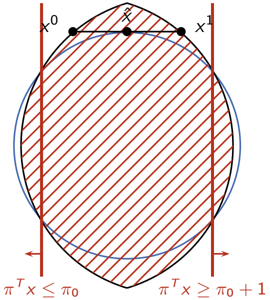

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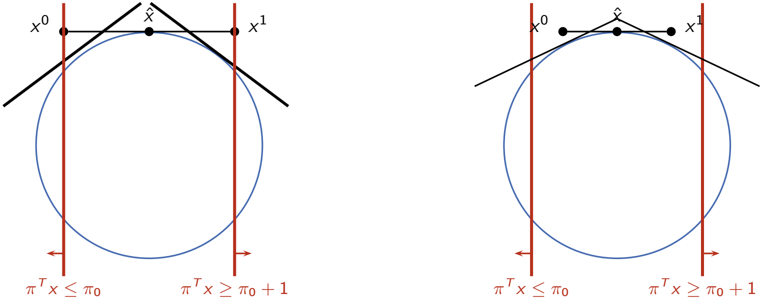

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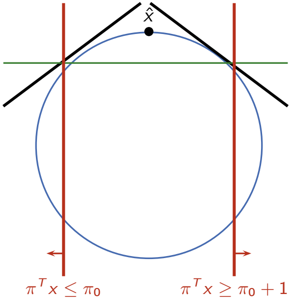

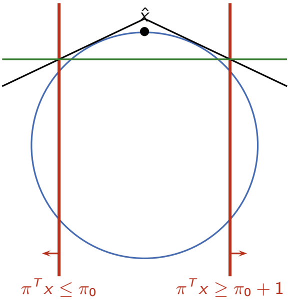

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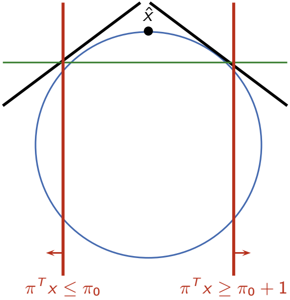

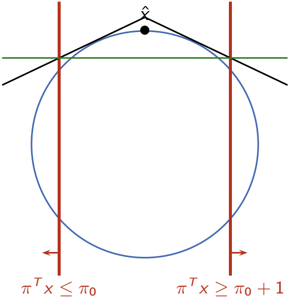

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Snapshot of results

- [Kılınc et al., 2011], report a speedup of 3 on a set of "hard" instances with the NLP/LP Filimint.
- [Bonami, 2011], report a speedup of 24 % on nontrivial instances with NLP BB.
- In both case, some instances not solved without these cuts are solved within seconds.

Combination with disaggregation[Kılınç, 2011]

Even better results are obtained by combining the extended formulation trick for separability and these cuts.

		Original		Extended	
	n	root gap	sol time	root gap	sol time
Batch	10	58.40	376.2	68.77	58.7
Markowitz	10	0.00	$>$ 10 800	98.07	1 262
SLay	14	68.50	36	86.08	5.0
uflquad	15	10.85	784	96.25	145

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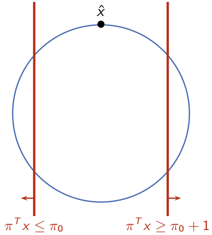
Sketch of an algorithm for MISOCP in CPLEX 12.6.2

- Only solve LPs,
- Dynamic generation of additional OA constraints.
- compact formulation using MLP,



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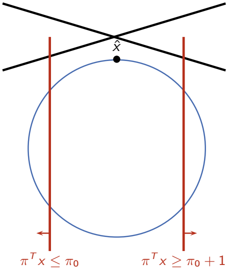

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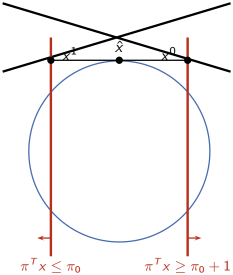

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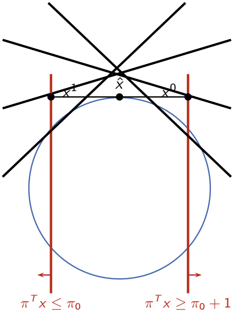

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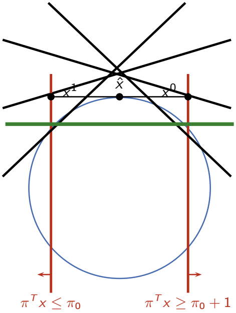

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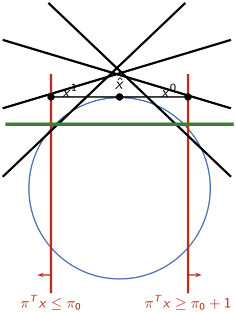

- Only solve LPs,
- Dynamic generation of additional OA constraints.
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Plus new recent features

- Propagation of conic constraints (12.6.1).
- Cone disaggregation for MISOCP (12.6.2).
- Redesigned heuristic choice of most promising algorithm (12.6.2).
- Improved OA Cuts using perspective reformulation (12.6.2).

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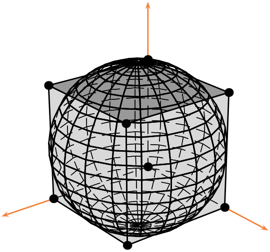
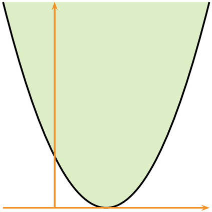
Putting it all together

[Cornuéjols and Li, 2001] showed that the empty ball in dimension n has *split rank n*
$\Rightarrow$ Practically unsolvable using any known cutting plane technique.



Instead the disaggregated formulation has (simple) split rank 1.



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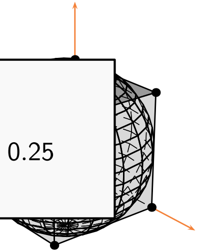
Putting it all together

[Cornuéjol
the empty
rank n
⇒ Practic
known cut

$$\left.\begin{array}{ll} -x_i + 0.25 \leq y_i & i = 1, \ldots, n \\ x_i - 0.75 + \leq y_i & i = 1, \ldots, n \\ x_i \leq 0 \text{ OR } x_i \geq 1 & \end{array}\right\} \Rightarrow y_i \geq 0.25$$



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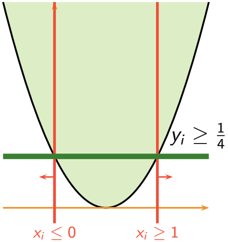

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The effect on our ellipse

$$\min \sum_{i=1}^{2n} i * x_i$$
$$\sum_{i=1}^{n}(100x_{2i}^2 + 100x_{2i-1}^2 - 4x_{2i}x_{2i-1} - 98x_{2i} - 98x_{2i-1}) \leq -1 \qquad (2)$$
$$x \in \mathbb{Z}^{2n}$$

- results on 12 threads with 12.6.1, 12.6.3[2], 12.6.3-- (no lift-and-project cuts) and 12.6.3++ (aggressive lift-and-project cuts), 3 hours time limit

	12.6.1	12.6.2--	12.6.2	12.6.2++
n	nodes	nodes	nodes	nodes
5	2,261	2,045	2,045	1,825
10	2,097,151	1,914,797	29	1
15	>23,125,426	>146,604,478	7,769	1

(Largest model solved in 2.2 sec by 12.6.3, in 5.5 sec by 12.6.2++.)

[2]Default results can be very sensitive to objective function

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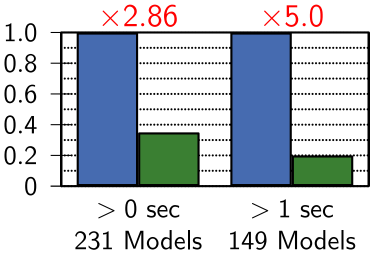
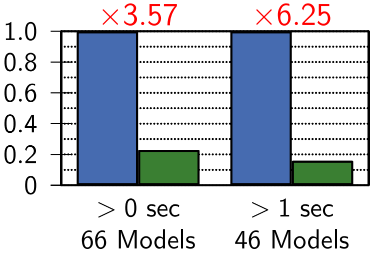
CPLEX 12.6.1 vs 12.6.3



CPLEX test bed

- CPLEX 12.6.1: 62 time limits
- CPLEX 12.6.3: 38 time limits



CBLIB

- CPLEX 12.6.1: 17 time limits
- CPLEX 12.6.3: 8 time limits

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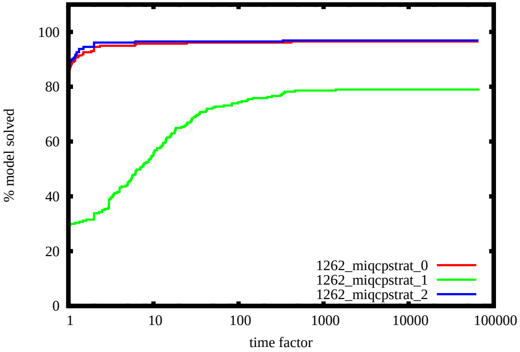
A comparison of OA and SOCP-BB in CPLEX 12.6.3 [3]



Default strategy picked

- OA 186 times
- SOCP-BB 4 times
- 55 models identical with both

To be perfect should have picked

- 2 more models with OA
- 9 more models with SOCP-BB

[3] 245 models solved by at least one method and failed by none.

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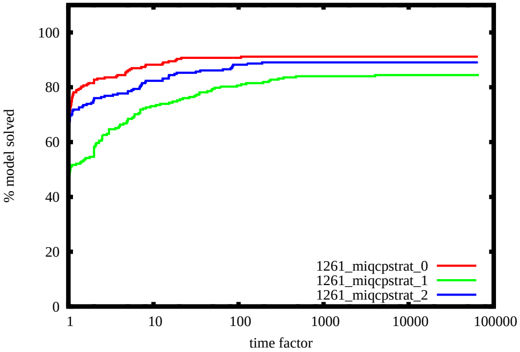
Reminder of CPLEX 12.6.1 [4]



Default strategy picked

- OA 113 times
- SOCP-BB 46 times
- 56 models identical with both

To be perfect should have picked

- 14 more models with OA
- 36 more models with SOCP-BB

[4] 225 models solved by at least one method and failed by none.

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Conclusion

- MINLP is still very challenging.
- Some significant applications solved but many still out of reach.
- According to S. Vigerske MINLP accelerates at a rate of 1.96/*year* (more than MILP's 1.8 claimed by B. Bixby).
- Conjecture: 25 years from now MINLPs will be solved 2.024×10^7 faster than today.
- Disciplined approach for MICP seems very promising.
- To get there we need:
  - more applications:
    `www.minlp.org`,
  - more benchmark instances:
    `www.gamsworld.org/minlp/minlplib2/html/`
  - more clever people!

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