Sparse Polynomial Optimization for Urban Distribution Networks

Martin Mevissen
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The Team

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Martin Mevissen  Nicole Taheri  Susara van den Heever  Rudi Verago
Motivation

- In many real-world decision problems, combined challenge of
  - Nonconvex models and dynamics, e.g. energy conservation laws (friction induced headloss, AC power flow).
  - Nonconvex objective functions, e.g. energy functionals, risk-averse optimization.
  - Uncertainty in problem parameters, e.g. demands, prices, supply.

- However, we do have
  - Samples of realizations for uncertain system parameters; e.g. collected iteratively by sensors and meters.
  - Constraints and decision variables highly structured, e.g. sparsity of traffic, energy or water networks.

- Optimal decision for these hard, nonconvex real-world problems in high demand!
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- Optimal decision for these hard, nonconvex real-world problems in high demand!
Motivation - AC Optimal Power Flow

Economic dispatch of power generation is a critical problem for utility companies,

Production Cost [O’Neill et al 2012]: \{ 519$bn Worldwide
112$bn USA

Goal: Determine the optimal operating point of an electric power generation system.

Challenges:
- non-convex due to the non-linear power flow equations
- lack of global solver for generic power systems

Approach:
- SDP provide strong bounds for ACOPF (Lavaei & Low 2010)
- Research on polynomial optimisation approach.

Benefits:
- Even 1% improvement in dispatch would result in 1-5$bn savings for US (4-20$bn worldwide) [O’Neill et al. 2012]
## A FERC Study: Where do Leading Solvers Fail?

<table>
<thead>
<tr>
<th>Termination</th>
<th>Conopt</th>
<th>Ipopt</th>
<th>Knitro</th>
<th>Minos</th>
<th>Snopt</th>
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</thead>
<tbody>
<tr>
<td>Exceeded time limit with an infeasible solution</td>
<td>7.9%</td>
<td>22.2%</td>
<td>38.4%</td>
<td>0.0%</td>
<td>23.0%</td>
</tr>
<tr>
<td>Exceeded time limit with a feasible solution</td>
<td>6.5%</td>
<td>0.9%</td>
<td>6.7%</td>
<td>0.0%</td>
<td>14.4%</td>
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<tr>
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<td>0.6%</td>
<td>11.4%</td>
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<td>72.5%</td>
<td>52.5%</td>
<td>30.3%</td>
<td>60.8%</td>
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Source: Anya Castillo and Richard O’Neill, Staff Technical Conference, FERC, Washington DC, June 24–26, 2013. 118–3120 bus instances, 100 runs per instance and solver, 20 min runs, Xeon E7458 2.4GHz, 64 GB RAM
Optimization over power systems:

- Large-scale of power transmission and distribution networks.
- AC power flow.
- Integration of distributed, uncertain renewable supply.
Pressure management in water distribution networks

Challenge: Leakage major problem in water distribution networks.

- **Variables:** Pressure $p_i$ at node $i$, and flow $q_{i,j}$ from node $i$ to $j$.
- **Parameters:** Elevation $e_i$ and uncertain demand $d_i$ at node $i$.
- **Decisions:** Setting $p_j$ at pressure reducing valves.
Pressure management in water distribution networks

Combination of challenges:

- Large-scale of water distribution networks
- Nonconvex friction-induced headloss
- Uncertain nodal demands.
Polynomial Optimization

**Polynomial Optimization Problem (POP)**

\[
\begin{align*}
\min & \quad p(x) \\
\text{s.t.} & \quad g_1(x) \geq 0, \ldots, g_m(x) \geq 0, \quad x = (x_1, \ldots, x_n)
\end{align*}
\]

**Hierarchy of SDP Relaxations for POP [Lasserre 2001]**

\[
\begin{align*}
d-\text{SDP}_\omega & \min \sum_\alpha p_\alpha y_\alpha \\
\text{s.t.} & \quad M_\omega(y) \succeq 0, \\
& \quad M_{\omega-\omega_j}(g_jy) \succeq 0, \quad j = 1, \ldots, m
\end{align*}
\]

If \( K \) compact and archimedean, \( \min(d-\text{SDP}_\omega) \to \min(\text{POP}) \) for \( \omega \to \infty \).

**Challenge:**

Size of the matrix inequalities \( \binom{n + \omega}{n} \) grows rapidly!
Polynomial Optimization

Polynomial Optimization Problem (POP)

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**d-SDP**

\[
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Exploiting structure for efficient solution methods

Decision optimization problem
Mathematical optimisation model

Structured Polynomial Optimization Problem
- Add valid inequalities to strengthen convexification.
- Exploit *correlative* sparsity of the POP.

SDP relaxation for POP
- Exploit *d- and r-space* sparsity in the SDP relaxation.
- Efficient algorithms for solving SDPs.
Exploiting structure for efficient solution methods

Decision optimization problem

Mathematical optimisation model

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<td></td>
<td><a href="#">Structured Polynomial Optimization Problem</a></td>
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</tr>
</tbody>
</table>
Hydraulic model for WDN

- Service requirements

\[
p_{\text{max}} \geq p_i \geq p_{\text{min}} \quad \forall i \in N,
\]
\[
q_{\text{max},i,j} \geq q_{i,j} \geq 0 \quad \forall (i,j) \in E.
\]

- Conservation of mass

\[
\sum_k q_{k,i} - \sum_l q_{i,l} = d_i \quad \forall i \in N.
\]

- Conservation of energy

If \( q_{i,j} > 0 \):

\[
p_i + e_i - p_j - e_j = h_{l_{i,j}}(q_{i,j}),
\]

If \( q_{i,j} = 0 \):

\[
p_i + e_i - p_j - e_j \leq 0,
\]

where \( h_{l_{i,j}}(q_{i,j}) \) the friction-induced head loss in pipe \((i,j)\) in case of flow from \(i\) to \(j\). Modeled as

\[
q_{i,j}(p_i + e_i - p_j - e_j - h_{l_{i,j}}(q_{i,j})) \geq 0,
\]
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p_i + e_i - p_j - e_j - h_{l_{i,j}}(q_{i,j}) \leq 0.
\]
Hydraulic model for WDN

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\[ p_{\text{max}} \geq p_i \geq p_{\text{min}} \quad \forall i \in N, \]
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- Conservation of energy

If \( q_{i,j} > 0 \): \( p_i + e_i - p_j - e_j = h_{l,i,j}(q_{i,j}), \)
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\[ q_{i,j}(p_i + e_i - p_j - e_j - h_{l,i,j}(q_{i,j})) \geq 0, \]
\[ p_i + e_i - p_j - e_j - h_{l,i,j}(q_{i,j}) \leq 0. \]
Modeling the friction loss

$h_{i,j}(q_{i,j})$ depends on roughness, diameter and length of $(i,j)$.

- Explicit formulation (Hazen-Williams):
  $$h_{i,j}(q_{i,j}) = c_{i,j}q_{i,j}^{1.852}$$

- Implicit formulation (Darcy-Weisbach):
  $$\begin{cases} h_{i,j}(q_{i,j}) = c_{i,j}^{l}q_{i,j} & \text{for laminar (i.e. small) flow regime,} \\ h_{i,j}(q_{i,j}) \approx c_{i,j}^{(2)}q_{i,j}^2 + c_{i,j}^{(1)}q_{i,j} & \text{for turbulent (i.e. large) flow regime} \end{cases}$$

- Piece-wise linear approximation of HW results in MILP.
- Global, quadratic lower and upper bounds of $h_{i,j}$ (HW). [Sherali & Smith 1997].
- Signpower for DW [Gleixner et al. 2011].
- Smooth approximation of Signpower HW [Bragalli et al. 2011].
- Our approach: Quadratic approximation of $h_{i,j}$ according to either HW or DW!
Modeling the friction loss

\[ h_{li,j}(q_{i,j}) \] depends on roughness, diameter and length of \((i,j)\).

- **Explicit formulation (Hazen-Williams):**
  \[ h_{li,j}(q_{i,j}) = c_{i,j}q_{i,j}^{1.852} \]

- **Implicit formulation (Darcy-Weisbach):**
  \[
  \begin{cases}
  h_{li,j}(q_{i,j}) = c_{i,j}q_{i,j}^1 & \text{for laminar (i.e. small) flow regime,} \\
  h_{li,j}(q_{i,j}) \approx c_{i,j}^{(2)} q_{i,j}^2 + c_{i,j}^{(1)} q_{i,j} & \text{for turbulent (i.e. large) flow regime}
  \end{cases}
  \]

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- **Our approach:** Quadratic approximation of \( h_{li,j} \) according to either HW or DW!
Quadratic fit of HW or turbulent flow DW model

• HW: Reasonable approx over range of flows.
• DW: If friction factor constant, headloss would be quadratic.
How is the approximation at low flows?

- Relative error is high, but absolute error is low
- Low flows were not problem in our tests

[B. Eck, M. Mevissen, Quadratic Approximations for Pipe Friction, J. Hydroinformatics, 2014]
Validation of fit in hydraulic model

Figure: Comparison of pressures and flows for the Exnet system as calculated by using quadratic headloss and EPANET 2.0
Cubic POP formulation for optimal valve setting

Derive the POP \((\text{VS-PP3})\)

\[
\begin{align*}
\min & \quad \sum_{i \in N} p_i \\
\text{s.t.} & \quad p_{\text{max}} \geq p_i \\
& \quad q_{\text{max},i,j} \geq q_{i,j} \\
& \quad \sum_k q_{k,i} - \sum_l q_{i,l} \\
& \quad q_{i,j}(p_i + e_i - p_j - e_j - h_{i,j}(q_{i,j})) \\
& \quad p_i + e_i - p_j - e_j - h_{i,j}(q_{i,j}) - M v_{i,j} \\
& \quad (1 - v_{j,i}) q_{\text{max},i,j} - q_{i,j} \\
& \quad v_{i,j} \in \{0, 1\} \\
\end{align*}
\]

where \(h_{i,j}(q_{i,j}) = a_{i,j} q_{i,j}^2 + b_{i,j} q_{i,j}\) and valve placement indicator \(v_{i,j} \in \{0, 1\}\) given for all \((i, j) \in E\).

[B. Eck, M. Mevissen, Fast non-linear optimization for design problems on water networks. World Environmental and Water Resources Congress 2013]
Quadratic POP formulation for optimal valve setting

Derive the POP \((\text{VS-PP2})\)

\[
\begin{align*}
\min & \quad \sum_{i \in N} p_i \\
\text{s.t.} & \quad p_{\text{max}} \geq p_i \geq p_{\text{min}} \quad \forall i \in N, \\
& \quad q_{\text{max},i,j} \geq q_{i,j} \geq -q_{\text{max},i,j} \quad \forall (i,j) \in E, \\
& \quad \sum_k q_{k,i} - \sum_l q_{i,l} = d_i \quad \forall i \in N, \\
& \quad p_j + e_j - p_i - e_i + h_{l,i,j}(q_{i,j})) = 0 \quad \forall (i,j) \in E, \text{if } v_{i,j} = 0, \\
& \quad v_{i,j} (p_j + e_j - p_i - e_i) + h_{l,i,j}(q_{i,j})) \leq 0 \quad \forall (i,j) \in E, \text{if } v_{i,j} \neq 0, \\
& \quad v_{i,j} q_{i,j} \geq 0 \quad \forall (i,j) \in E, \\
& \quad v_{i,j} \in \{-1, 0, 1\}
\end{align*}
\]

where \(h_{l,i,j}(q_{i,j}) = a_{i,j} q_{i,j} |q_{i,j}| + b_{i,j} q_{i,j}\), valve placement indicator \(v_{i,j} \in \{0, 1\}\) given for all \((i,j) \in E\) and \(q_{i,j} \in \mathbb{R}\).

New quadratic model (with Mathieu Claeys, Bissan Ghaddar).
Exploit correlative sparsity [Waki et al. 2006]

\[
\begin{align*}
\text{min} & \quad \sum_{i=1}^{6} x_i \\
\text{s.t.} & \quad x_i^2 \leq 2 \quad \forall i \\
& \quad x_1 + x_6 \geq 1, \quad x_2 - x_6 \leq 1, \quad x_3 + x_4 \geq 1, \\
& \quad x_3 - x_6 \leq 1, \quad x_4 + x_5 \geq 1, \quad x_5 - x_6 \leq 1.
\end{align*}
\]

(a) \( C_1 = \{3, 4, 6\} \), \( C_2 = \{4, 5, 6\} \), \( C_3 = \{1, 6\} \), \( C_4 = \{2, 6\} \)
Sparse hierarchy

Hierarchy of sparse SDP Relaxations for POP [Waki et al. 2006]

\[
s\text{-SDP}_\omega \quad \min \sum_\alpha p_\alpha y_\alpha \\
\text{s.t.} \quad M_\omega(y, C_k) \succeq 0, \quad k = 1, \ldots, P \\
M_{\omega-\omega_j}(g_j y, C_k) \succeq 0, \quad j \in J_k, k = 1, \ldots, P
\]

\[
\min(\text{s-SDP}_\omega) \leq \min(\text{d-SDP}_\omega)
\]

If \( K \) compact, archimedean, and \((C_k)_{k=1}^P\) satisfy the running-intersection property, \( \min(\text{s-SDP}_\omega) \to \min(\text{POP}) \) for \( \omega \to \infty \). [Lasserre 2006]

Reduction in size:

Size of \( k \) matrix inequalities \( \begin{pmatrix} C_k & +\omega \\ C_k & \end{pmatrix} \) vastly smaller if \( |C_k| \ll n \).
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**Reduction in size:**

Size of \( k \) matrix inequalities \( \begin{bmatrix} |C_k| & +\omega \\ |C_k| & \end{bmatrix} \) vastly smaller if \(|C_k| \ll n\).
Valve setting POP satisfies structured sparsity:

nz = 8880

nz = 15165

Pescara: $n = 270$, $P = 210$, $3 \leq |C_k| \leq 12$. 
Sparse SDP relaxations for (VS-PP3) - Pescara

[Bragalli et al. 2011]

<table>
<thead>
<tr>
<th>PRV to set</th>
<th>sSDP₂ Lbd</th>
<th>SDPA Time</th>
<th>IPOPT Obj</th>
<th>Gap</th>
<th>IPOPT Time</th>
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Sparse SDP relaxations for \((\text{VS-PP2})\) - Pescara

[Bragalli et al. 2011]

<table>
<thead>
<tr>
<th>PRV</th>
<th>SeDuMi sSDP_1</th>
<th>Obj</th>
<th>Time</th>
<th>FE</th>
<th>sSDP_2</th>
<th>Obj</th>
<th>Time</th>
<th>FE</th>
<th>SDPA sSDP_1</th>
<th>Obj</th>
<th>Time</th>
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<th>sSDP_2</th>
<th>Obj</th>
<th>Time</th>
<th>FE</th>
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<tr>
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<td></td>
<td>1657</td>
<td>384</td>
<td>0.02</td>
<td></td>
</tr>
</tbody>
</table>
Optimal Power Flow Problem

Rectangular Power-Voltage ACOPF

\[
\min \sum_{k \in g} \left( c_k^2 (P_k^g)^2 + c_k^1 P_k^g + c_k^0 \right)
\]

\[
P_k + j Q_k = V_k (Y_k^* V_k^*)
\]

\[
P_k^{\min} \leq P_k^g \leq P_k^{\max}, \quad Q_k^{\min} \leq Q_k^g \leq Q_k^{\max}
\]

\[
(V_k^{\min})^2 \leq \Re V_k^2 + \Im V_k^2 \leq (V_k^{\max})^2 \quad \forall k \in N
\]

\[
P_{lm}^2 + Q_{lm}^2 \leq S_{l,m}^{\max} \quad \forall (l, m) \in L
\]
Optimal Power Flow Problem

**POP formulation for ACOPF (OPF-PP4)**

\[
\min \sum_{k \in G} \left( c_k^2 (P_k^d + \text{tr}(Y_k xx^T))^2 + c_k^1 (P_k^d + \text{tr}(Y_k xx^T)) + c_k^0 \right)
\]

\[
P_k^{\text{min}} - P_k^d \leq \text{tr}(Y_k xx^T) \leq P_k^{\text{max}} - P_k^d \quad \forall i \in N
\]

\[
Q_k^{\text{min}} - Q_k^d \leq \text{tr}(\bar{Y}_k xx^T) \leq Q_k^{\text{max}} - Q_k^d \quad \forall i \in N
\]

\[
(V_k^{\text{min}})^2 \leq \text{tr}(M_k xx^T) \leq (V_k^{\text{max}})^2 \quad \forall i \in N
\]

\[
(\text{tr}(Y_{lm} xx^T))^2 + (\text{tr}(\bar{Y}_{lm} xx^T))^2 \leq (S_{lm}^{\text{max}})^2 \quad \forall (l, m) \in L
\]

dSDP_2 : [Josz et al. 2013], [Molzahn et al. 2014]
Optimal Power Flow Problem

**POP formulation for ACOPF (OPF-PP2)**

\[
\begin{align*}
\min & \quad \sum_{k \in G} \left( c_k^2(P_g^k)^2 + c_k^1(P_d^k + \text{tr}(Y_kxx^T)) + c_k^0 \right) \\
\text{s.t.} & \quad P_{k}^{\text{min}} \leq \text{tr}(Y_kxx^T) + P_{k}^{d} \leq P_{k}^{\text{max}} \\
& \quad Q_{k}^{\text{min}} \leq \text{tr}(\bar{Y}_kxx^T) + Q_{k}^{d} \leq Q_{k}^{\text{max}} \\
& \quad (V_{k}^{\text{min}})^2 \leq \text{tr}(M_kxx^T) \leq (V_{k}^{\text{max}})^2 \\
& \quad P_{lm}^2 + Q_{lm}^2 \leq (S_{lm}^{\text{max}})^2 \\
& \quad P_{g}^k = \text{tr}(Y_kxx^T) + P_{k}^{d} \\
& \quad P_{lm} = \text{tr}(Y_{lm}xx^T) \\
& \quad Q_{lm} = \text{tr}(\bar{Y}_{lm}xx^T)
\end{align*}
\]

sSDP$$_1$$ and dSDP$$_1$$ equivalent to [Lavaei, Low 2010]
Moment approaches for (OPF-PP4)

Limitations of Lavei & Low:
[Lesieutre et al. 2011]; [Bukhsh et al. 2012]

Dense Lasserre hierarchy for (OPF-PP4)
[Josz et al. (RTE France) 2013], [Molzahn et al 2013]
Solving feeders up to 10 buses to global optimality.

Prominent instances: WB2, LMBM3, WB5, ..
Exploit sparsity in matrix inequalities [Kim et al. 2011]

\[
\begin{align*}
\min A^0 \bullet X \quad \text{s.t.} \quad & M(X) \succeq 0, \ X \succeq 0, \\
\text{where} \ A^0 \in S^n \text{ tridiagonal}, \ X \in S^n, \ \text{and} \\
M(X) &= \begin{pmatrix}
1 - X_{11} & 0 & 0 & \cdots & 0 & X_{12} \\
0 & 1 - X_{22} & 0 & \cdots & 0 & X_{23} \\
0 & 0 & \ddots & \ddots & \ddots & \ddots \\
\cdots & \cdots & \cdots & \ddots & \ddots & \ddots \\
0 & 0 & 0 & \cdots & 1 - X_{n-1,n-1} & X_{n-1,n} \\
X_{21} & X_{32} & X_{43} & \cdots & X_{n,n-1} & 1 - X_{nn}
\end{pmatrix}.
\end{align*}
\]

- Only $X_{i,j}$ with $|i - j| \leq 1$ are relevant for evaluating objective or matrix inequality constraint.
- The matrix inequality satisfies some sparsity pattern as well.
Sparsity pattern for the SDP example

- **Domain-space sparsity:**

\[
\begin{pmatrix}
\ast & \ast & 0 & \ldots & 0 & 0 \\
\ast & \ast & \ast & \ldots & 0 & 0 \\
0 & \ast & \ast & \ldots & 0 & 0 \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
0 & 0 & \ldots & \ldots & \ast & \ast \\
0 & 0 & \ldots & \ldots & \ast & \ast \\
\end{pmatrix}
\]

- **Range-space sparsity:**

\[
\begin{pmatrix}
\ast & 0 & \ldots & 0 & \ast \\
0 & \ast & \ldots & 0 & \ast \\
\ldots & \ldots & \ldots & \ldots & \ldots \\
0 & 0 & \ldots & \ast & \ast \\
\ast & \ast & \ldots & \ast & \ast \\
\end{pmatrix}
\]
Theorem [Grone 1984]

Let $C_k (k = 1, 2, \ldots, p)$ be the maximal cliques of a chordal graph $G(N, E)$, $N \in \{1, \ldots, n\}$. Suppose that $X \in S^n(E, ?)$. Then

$$X \in S^n_+(E, ?) \iff X(C_k) \in S^C_{+k} (k = 1, \ldots, p).$$

(a) $C_1 = \{3, 4, 6\}$, $C_2 = \{4, 5, 6\}$, $C_3 = \{1, 6\}$, $C_4 = \{2, 6\}$
Applying d-space and r-space conversion yields:

**Dense SDP**

\[
\min \ A^0 \bullet X \quad \text{s.t.} \quad M(X) \succeq 0, \ X \succeq 0
\]

**Equivalent reduced SDP using PSD Completion [Grone 1984]**

\[
\min \ \sum_{i=1}^{n-1} (A_{ii}^0 X_{ii} + 2A_{i,i+1}^0 X_{i,i+1}) + A_{nn}^0 X_{nn}
\]

\[
\begin{align*}
\text{s.t.} \quad & \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} X_{11} & -X_{12} \\ -X_{21} & -z_1 \end{pmatrix} \succeq 0, \\
& \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} X_{ii} & -X_{i,i+1} \\ -X_{i+1,i} & z_{i-1} - z_i \end{pmatrix} \succeq 0 \ (i = 2, 3, \ldots, n - 2), \\
& \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} X_{n-1,n-1} & -X_{n-1,n} \\ -X_{n,n-1} & X_{n,n} + z_{n-2} \end{pmatrix} \succeq 0, \\
& \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} -X_{ii} & -X_{i,i+1} \\ -X_{i+1,i} & -X_{i+1,i+1} \end{pmatrix} \succeq 0 \ (i = 1, 2, \ldots, n - 1).
\end{align*}
\]
Sparsity of the rank relaxation due to Lavaei-Low

- Sparsity of admittance matrix can be exploited. [Stott 1974].
- Exploit sparsity in SDP relaxation for OPF [Molzahn et al. 2013].

![Graph](image)

<table>
<thead>
<tr>
<th>n</th>
<th>Obj</th>
<th>Dual of dSDP₁((OPF-PP2))</th>
<th>Dim</th>
<th>Time(Original)</th>
<th>Time(SparseCoLO)</th>
</tr>
</thead>
<tbody>
<tr>
<td>14</td>
<td>8081.52</td>
<td>875×90</td>
<td>0.96</td>
<td>0.44</td>
<td></td>
</tr>
<tr>
<td>30</td>
<td>574.52</td>
<td>4500×666</td>
<td>2.26</td>
<td>2.04</td>
<td></td>
</tr>
<tr>
<td>118</td>
<td>129654.6</td>
<td>56546×806</td>
<td>9.26</td>
<td>6.18</td>
<td></td>
</tr>
<tr>
<td>300</td>
<td>719724.7</td>
<td>361683×1776</td>
<td>94.03</td>
<td>13.66</td>
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</tr>
<tr>
<td>2736sp</td>
<td>1.307×10⁶</td>
<td>30019740×57408</td>
<td>*</td>
<td>3502.2</td>
<td></td>
</tr>
</tbody>
</table>
Dense Vs. sparse hierarchy - LMBM3

Table: Second order relaxation for LMBM3, where first order not exact.

<table>
<thead>
<tr>
<th>$S_{23}^{\max}$</th>
<th>dSDP$_2$(OPF-PP4)</th>
<th>sSDP$_2$(OPF-PP4)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bound</td>
<td>Time</td>
<td>Bound</td>
</tr>
<tr>
<td>28.35</td>
<td>10294.88</td>
<td>10294.88</td>
</tr>
<tr>
<td>31.16</td>
<td>8179.99</td>
<td>8179.99</td>
</tr>
<tr>
<td>33.96</td>
<td>7414.94</td>
<td>7414.94</td>
</tr>
<tr>
<td>36.77</td>
<td>6895.19</td>
<td>6895.19</td>
</tr>
<tr>
<td>39.57</td>
<td>6516.17</td>
<td>6516.17</td>
</tr>
<tr>
<td>42.38</td>
<td>6233.31</td>
<td>6233.31</td>
</tr>
<tr>
<td>45.18</td>
<td>6027.07</td>
<td>6027.07</td>
</tr>
<tr>
<td>47.99</td>
<td>5882.67</td>
<td>5882.67</td>
</tr>
<tr>
<td>50.79</td>
<td>5792.02</td>
<td>5792.02</td>
</tr>
<tr>
<td>53.60</td>
<td>5745.04</td>
<td>5745.04</td>
</tr>
</tbody>
</table>

But: For $> 10$ buses, dense hierarchy intractable.
Dense Vs. sparse hierarchy - LMBM3

Table: Second order relaxation for LMBM3, where first order not exact.

<table>
<thead>
<tr>
<th>$S_{23}^{\text{max}}$</th>
<th>$d\text{SDP}_2(\text{OPF-PP4})$</th>
<th>$s\text{SDP}_2(\text{OPF-PP4})$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Bound Time</td>
<td>Bound Time</td>
</tr>
<tr>
<td>28.35</td>
<td>10294.88 1.2</td>
<td>10294.88 1.0</td>
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<tr>
<td>31.16</td>
<td>8179.99 0.8</td>
<td>8179.99 0.7</td>
</tr>
<tr>
<td>33.96</td>
<td>7414.94 1.0</td>
<td>7414.94 0.8</td>
</tr>
<tr>
<td>36.77</td>
<td>6895.19 0.8</td>
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<td>42.38</td>
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<td>6027.07 0.9</td>
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<td>5745.04 0.8</td>
</tr>
</tbody>
</table>

But: For $> 10$ buses, dense hierarchy intractable.
Exploiting sparsity in ACOPF and its SDP relaxation

<table>
<thead>
<tr>
<th>N</th>
<th>dual of dSDP$_1$(OPF-PP2)</th>
<th>primal of sSDP$_2$(OPF-PP4)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Bound</td>
<td>Time</td>
</tr>
<tr>
<td>9mod</td>
<td>2753.2</td>
<td>0.61</td>
</tr>
<tr>
<td>14mod</td>
<td>7792.7</td>
<td>0.93</td>
</tr>
<tr>
<td>30mod</td>
<td>576.9</td>
<td>3.81</td>
</tr>
<tr>
<td>39</td>
<td>41862.1</td>
<td>2.20</td>
</tr>
<tr>
<td>57</td>
<td>41737.8</td>
<td>3.2</td>
</tr>
<tr>
<td>118</td>
<td>129654.6</td>
<td>6.18</td>
</tr>
<tr>
<td>300</td>
<td>719724.7</td>
<td>13.66</td>
</tr>
<tr>
<td>2736sp</td>
<td>1.307x10$^6$</td>
<td>3502.2</td>
</tr>
</tbody>
</table>

Exploiting correlative sparsity of (OPF-PP4) allows to find global optima for instances with up to 40 buses.

Conclusion and open questions

- Exploiting problem structure (sparsity, symmetry, decomposition ...) in industrial decision optimisation problems crucial.
- Techniques available to scale-up powerful polynomial optimisation approaches.
- Combine advances in real algebra, polynomial optimisation and algorithms for semidefinite programs.

Thanks for your attention!
Conclusion and open questions

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- Techniques available to scale-up powerful polynomial optimisation approaches.
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Thanks for your attention!
B. Eck, M. Mevissen, Quadratic Approximations for Pipe Friction, J. Hydroinformatics, 2014, To Appear

B. Eck, M. Mevissen, Fast non-linear optimization for design problems on water networks. World Environmental and Water Resources Congress 2013